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Technical and conceptual aspects in the proof of the a-theorem

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Introduction

One of the main advancements in theoretical physics of the 20th century is the idea of the renormalization group (RG). At its core, the idea is very simple. It addresses the question of the role of scale in any problem. What happens when we look at something from different scales? Do we need to know completely the physics at small distances to be able to infer results about long-range physics? It is exactly this question that the RG tries to answer. One starts with a microscopic theory and then zoom out or coarse grain. The hope is that if we are interested in long-range universal physics, we can by the procedure of the RG essentially eliminate all the details which are irrelevant at long distances and arrive at a more simple but effective description of the long-range physics we are interested in. It is effective in the sense that it will only give accurate results down to a certain scale after which short-range effects will start playing a role again.

The above description can essentially be summarized in the following way: The RG is the study of the change of dynamics of a theory under scale change or dilation. We have just argued that under the RG, we start from a more complicated theory and end up at an (ideally) more simple one. Very generally, we can therefore say that the RG induces a motion in the space of all theories, from one theory to another. This motion is often called the RG flow. The above heuristic way of describing the RG makes the basic intuition we have about what this process should do very clear: In the process of zooming out, we are eliminating or removing degrees of freedom, so that in some sense, we are tacitly assuming the irreversibility of the RG flow. Irreversibility not in the thermodynamic sense of independence of the initial conditions, but irreversibility in the sense that if we start from some theory and start moving along the RG flow, we can never end up at the same theory again. Of course, this is an assumption, which, if true, puts constraints on the physics we are describing, but it is not a priori guaranteed that this holds in the formal implementation of the RG.

Now enters the a-theorem (or c-theorem). In 1986, Zamolodchikov[1] did the first step towards answering this question in what is called today Zamolodchikov's theorem or the c-theorem. He proved that for 2 dimensional unitary field theories, there exists a function C that changes monotonously along the RG flow, thereby establishing the irreversibility of the process. Shortly after, in 1988, Cardy[2] conjectured the existence of a higher-dimensional analog of the c-theorem, called the a-theorem. Even though a lot of evidence was accumulated over the years and a proof to first order in perturbation theory was given by Jack and Osborn in 1990[3], a non-perturbative proof was still missing. In 2011, a very convincing attempt at a proof was given in the work of Komargodski and Schwimmer[4] and Komargodski[5]. This was then refined and extended by Luty, Polchinski and Rattazzi[6] and improved further by Baume, Keren-Zur, Rattazzi and Vitale[7]. The idea of Komargodski and Schwimmer was to study a specific combination of correlators of the trace of the energy-momentum tensor, packaged as the scattering amplitude of a background field called the dilaton, and then obtain positivity constraints

on the latter from unitarity by using a dispersion relation. Let us emphasize at this point that unitarity is one of the main assumptions in the proof. For some of the results, there exist counterexamples if one considers non-unitary theories.

This thesis is about some technical and conceptual aspects in the proof of the a-theorem and it will be organized as follows. The first chapter of this thesis deals with the techniques involved in the proofs of the a-theorem and constraints on RG flows. We start with a section in two dimensions. The reader is strongly encouraged to read this section for several reasons. First of all, it contains all the techniques which enter in the proof of the a-theorem. There will not be anything essentially new when we will deal with the four dimensional case. The second reason is that in two dimensions, the computations, while still substantial, are by far not as cumbersome as in four dimensions. They are presented in all detail in the two dimensional case, while in four dimensions, we will only report the results of interest. The organization of both sections is very similar. We start by studying the local RG equation[8, 3, 7]. It is a powerful technique, which enables one to perturbatively study composite operators and their properties under RG flows. By adding certain consistency conditions, one can then deduce highly non-trivial constraints on quantities of interest such as beta functions. We then turn to the dilaton effective action, which very roughly is an efficient way to study correlators of the trace of the energy-momentum tensor off-criticality, i.e. when we perturbatively move away from the conformal fixed point. We then finally turn towards using a quantity known as the dilaton scattering amplitude to prove some highly non-trivial results, namely the non-perturbative a-theorem and the fact that under very broad conditions a QFT always asymptotes (under the RG flow) to a CFT.

The second chapter deals with possible issues in the proof and starts to explore solutions to those. This was the original goal and motivation of this thesis. It needs to be emphasized that no solutions have been found and this chapter does not contain any significant results. It is merely a report of the various paths that have been explored by the author which have not yielded any real insights into solutions of the problems. This chapter starts by clearly stating where problems could arise in the proof. The next sections suggest some solution avenues.

In the various sections, the reader will find parts of the text which are separated from the rest by a line of asterisks and the fact that they are written in italic. These parts in general contain the more computational parts of the presented developments. We have chosen this way of organizing the text because we feel that these computations are an integral part of understanding what is being done and putting them into the appendix would break the logical continuity of the developments. We also realize that this is not everybody's opinion and maybe more importantly, one might only be interested in the main points and results, which is why we have come to the above mentioned compromise. We also want to emphasize that in this thesis, $\langle \dots \rangle$ is always going to represent *time-ordered* correlators, except if explicitly mentioned otherwise.

Chapter 1

The a-theorem and its proof

1.1 The a-theorem(s)

There are two essentially two versions of the a-theorem and one important refinement.

Theorem (Weak a-theorem). *Assume that we have an RG flow between two conformal field theories (CFT). Then there exist two numbers a_{UV} and a_{IR} , which characterize each conformal fixed point and which satisfy*

$$a_{UV} \geq a_{IR} \quad (1.1.1)$$

Theorem (Strong a-theorem). *There exists a function $\tilde{a}$ which increases monotonously along the RG flow as the renormalization scale μ is increased*

Theorem (Gradient flow formula). *The RG evolution of $\tilde{a}$ is characterized by a strong increase condition:*

$$\partial_I \tilde{a} = \chi_{IJ} B^J \quad (1.1.2)$$

where ∂_I represents the derivative with respect to the arguments of $\tilde{a}$ which are the couplings, χ_{IJ} is a positive definite matrix and B^K is the non-ambiguous beta function, which we will define later.

The weak a-theorem already implies a notion of irreversibility. Indeed, it excludes for instance a cyclic behavior of the RG flow. If we continue to flow after the first IR theory and attain a deeper IR fixed point, the theorem implies that its a coefficient should be smaller than a_{IR} and the new fixed point therefore can not be the same as the the initial one. It also gives the possibility to exclude certain RG flows: For example, if you have an IR theory, you can exclude all theories with a lower anomaly coefficient as the UV theories from which the flow started. The weak a-theorem is also the only theorem for which we can hope to find a perturbative proof, because it only relies on information about the fixed points.

The strong a-theorem implies a stronger notion of irreversibility because it excludes any strange behavior of the RG flow between the two fixed points. Ideally, one could hope for a non-perturbative proof of the strong a-theorem, but it seems rather unlikely because generally along an RG flow the theory goes through a non-perturbative regime about which one cannot say much. The proof(s) we give in this work are perturbative.

Finally, the gradient flow formula is the strongest statement, which also is inherently perturbative since it is formulated using beta functions. From the gradient flow formula,

one can prove other types of results about scale and conformal invariance and the asymptotics of quantum field theories about which we will have more to say later on. Notice also that the gradient flow formula implies the strong a-theorem if one contracts it with B^I , which also shows the importance of the positive definiteness of χ_{IJ} ¹.

This chapter is organized as follows. We will start by discussing the a-theorem in two dimensions, because it is an appropriate playing ground to introduce the different techniques and concepts which are used in the subsequent chapter about the four dimensional a-theorem. We discuss the simplest form of local RG equation. We then introduce the dilaton into the mix and use it to prove a non-perturbative version of the weak a-theorem as well as Zamolodchikov's theorem. Along the way, we will also introduce some key concepts such as the trace anomaly and the dilaton Wess-Zumino action.

In the second part of chapter, the discussion will be essentially the same except that we now study the four dimensional case. We discuss the local RG equation as an efficient tool to study properties of the RG flow. We then present the non-perturbative argument for the weak a-theorem and the perturbative argument for the strong a-theorem. Finally, we will discuss how one can use these methods to constrain the asymptotics of QFTs, giving a perturbative proof that they are necessarily conformal.

1.2 Introducing the tools in 2D

1.2.1 Simplest form the local RG equation

The idea of promoting sources to background fields chosen to transform in a specific way under a symmetry so as to compensate the breaking of that symmetry in the original theory has been known and used extensively for more than 30 years. One instance where this idea appears is that one possible interpretation of the Callan-Symanzik equation is to see it as a consequence of exploiting the enhanced symmetry coming from promoting the coupling constants to background fields whose transformation under dilation is controlled by the β function and the anomalous dimension so as to compensate for the explicit breaking of scale symmetry by dimensionful parameters and the dependence on the renormalization scale. The reason this approach is so useful is because the thereby generated enhanced symmetry allows stronger constraints on the form of the resulting effective action and better control over the correlators of the composite operators sourced by the background fields.

The local RG equation is simply this idea pushed a bit further: The symmetry is taken to be local. Since we are interested in general properties of QFTs under dilations which are controlled by correlators of the energy-momentum (EM) tensor and its trace, it is logical to add a source term to a theory to generate these. But it is well known that the EM tensor is sourced by the metric, which is why we will make our considerations in a background metric, where the local scale is going to be realized by Weyl symmetry. It is worth noting at this point that our interest in the correlators of the trace of the EM tensor is an early indication as to why later on we are going to work with the dilaton since the latter is essentially a source for the trace of the EM tensor.

1.2.1.1 Set-up

We want to study the RG flow in the neighborhood of a conformal invariant fixed point. As already mentioned previously, the idea is to deform the theory by introducing interactions which explicitly break conformal invariance and to restore the latter symmetry by assigning

¹Notice that at this stage, it is not clear why we want positive definiteness and not negative definiteness.

the appropriate transformations to the corresponding background fields. In this first section, we will consider only *marginal* deformations of the fixed point, i.e. dimension 2 operators and ignore all subtleties arising from other types of deformations (either from currents or operators of dimension less than 2). We will discuss the effects of these other types of deformations in more detail in the four dimensional case.

We now turn on the marginal deformations described by the couplings λ^I where $I = 1, \dots, N$ which are associated to the scalar operators $\mathcal{O}_I$. We therefore have a set of sources we collectively call $\mathcal{J} = (g_{\mu\nu}, \lambda^I)$. We are going to denote setting the source to their fixed point value as $\mathcal{J} = 0$ which simply means taking $g_{\mu\nu} = \eta_{\mu\nu}$ and $\lambda^I = \text{const.}$

We then define the theory by its partition function:

$$\mathcal{Z}[\mathcal{J}] = e^{i\mathcal{W}[\mathcal{J}]} = \int \mathcal{D}[\Phi] e^{iS[\Phi, g] + i \int d^2x \sqrt{-g} \lambda \cdot \mathcal{O}} \quad (1.2.1)$$

where S is the action of the underlying theory. The crucial assumption here is that the partition function is *renormalized*, meaning that it generates the correlators of *renormalized* composite operators. This assumption is critical for everything that follows and sensible since we know that the renormalization of composite operators can be done in a consistent way (see for instance [9]). Another advantage is that we have not specified the regulator nor the renormalization procedure and we are therefore a priori independent of these choices.

We now recall the standard definition of the EM tensor:

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}} \frac{\delta S}{\delta g^{\mu\nu}(x)} \quad (1.2.2)$$

As always, the time-ordered correlators are then given by functional derivatives of the corresponding sources:

$$\begin{aligned} \langle T_{\mu\nu}(x) \rangle &= \frac{1}{\mathcal{Z}[\mathcal{J} = 0]} \int \mathcal{D}[\Phi] T_{\mu\nu}(x) e^{iS[\Phi, \eta]} \\ &= \frac{1}{\mathcal{Z}[\mathcal{J} = 0]} \int \mathcal{D}[\Phi] \frac{2}{\sqrt{-g(x)}} \frac{\delta S}{\delta g^{\mu\nu}(x)} e^{iS[\Phi, g]} \Big|_{g=\eta} \\ &= \frac{2}{i\sqrt{-g(x)}} \frac{1}{\mathcal{Z}[\mathcal{J} = 0]} \frac{\delta \mathcal{Z}[\mathcal{J}]}{\delta g^{\mu\nu}(x)} \Big|_{\mathcal{J}=0} \\ &= \frac{2}{\sqrt{-g(x)}} \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta g^{\mu\nu}(x)} \Big|_{\mathcal{J}=0} \end{aligned}$$

So that in particular, we also have:

$$\langle T(x) \rangle = \frac{2}{\sqrt{-g(x)}} g^{\mu\nu}(x) \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta g^{\mu\nu}(x)} \Big|_{\mathcal{J}=0} \quad (1.2.3)$$

where T denotes the trace of the EM tensor.

More generally, we now *define*:

$$\begin{aligned} \langle T(x_1) \cdots T(x_n) \rangle &= \frac{(-i)^{n-1} 2^n}{\sqrt{-g(x_1)} \cdots \sqrt{-g(x_n)}} g^{\mu_1 \nu_1}(x_1) \frac{\delta}{\delta g^{\mu_1 \nu_1}(x_1)} \\ &\quad \times g^{\mu_2 \nu_2}(x_2) \frac{\delta}{\delta g^{\mu_2 \nu_2}(x_2)} \cdots g^{\mu_n \nu_n}(x_n) \frac{\delta}{\delta g^{\mu_n \nu_n}(x_n)} \mathcal{W}[\mathcal{J}] \Big|_{\mathcal{J}=0} \end{aligned} \quad (1.2.4)$$

By defining the correlators of T in this way, we simplify our life later on at the expense of these correlators differing from the standard ones by contact terms.

In the same way we have:

$$\langle \mathcal{O}_{I_1}(x_1) \cdots \mathcal{O}_{I_n}(x_n) \rangle = \frac{(-i)^{n-1}}{\sqrt{-g(x_1)} \cdots \sqrt{-g(x_n)}} \frac{\delta}{\delta \lambda^{I_1}(x_1)} \cdots \frac{\delta}{\delta \lambda^{I_n}(x_n)} \mathcal{W}[\mathcal{J}] \Big|_{\mathcal{J}=0} \quad (1.2.5)$$

It is important to keep in mind that although we have not indicated it explicitly the sources and their corresponding operators depend on a renormalization scale μ . Furthermore, we want to emphasize that this technique of taking derivatives of the generating functional generates the *time-ordered* correlators and as such $\langle \cdots \rangle$ will always represent time-ordered correlators, unless explicitly stated otherwise.

The point of all of this is that the effective action for the sources formally respects the enhanced symmetries, up to one caveat which turns out to be the crucial ingredient in all of what follows. Indeed, it is a well-known fact of QFT that the renormalization procedure more often than not introduces effects which will break a classical symmetry at the quantum level. This breaking of symmetry at the quantum level is represented by contact terms called *anomalies* in the Ward identities of the corresponding symmetry. We will turn to these in a moment. By introducing a background metric, we introduce diffeomorphism invariance as one the extended symmetries of the effective action. For the rest of this work, we are going to assume that diffeomorphisms are anomaly free.

As already stated, another symmetry we are interested in is Weyl symmetry² under which the metric transforms as

$$g^{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g^{\mu\nu}(x) \quad \& \quad \delta_\sigma g^{\mu\nu}(x) = 2\sigma(x) g^{\mu\nu}(x) \quad (1.2.6)$$

It turns out that Weyl symmetry is anomalous in curved space. Its anomaly is the most important ingredient in all that follows. How and why this anomaly arises can be seen very explicitly in for instance dimensional regularization. See for instance [10] for a review and further references. Here we are only going to give a basic intuitive idea of how this works (based on the argument in [7]). It is always possible to split the bare action into two parts:

$$S[\Phi, \mathcal{J}] = S^{(1)}[\Phi, \mathcal{J}] + S^{(2)}[\mathcal{J}] \quad (1.2.7)$$

where all the non-trivial terms depending on the dynamical fields are contained in $S^{(1)}$ whereas $S^{(2)}$ only contains terms depending on the sources. This is the well-known procedure of renormalized perturbation theory which is necessary to satisfy the original assumption we made which is that the effective action generates the correlators of the renormalized operators. Once again, the basis of the method we are pursuing here is to assign transformations to sources so as to compensate the explicit breaking of symmetries, which effectively means that we can choose a Weyl transformation of the sources $\delta_\sigma \mathcal{J}$ such that $S^{(1)}$ is invariant under Weyl transformation. The fact that one can do this can for example be seen very clearly in dimensional regularization (see Appendix B in [7]), but maybe more relevantly the assumption that we can do this is the basic assumption from which we start. Of course, once we have fixed this transformation of the sources, there is nothing which guarantees that $S^{(2)}$ is going to be invariant under this form of Weyl transformations and in general it is not. We therefore have obtained that the Weyl variation of the effective action is completely due to the variation of $S^{(2)}$:

$$\int d^D x \delta_\sigma \mathcal{J} \frac{\delta}{\delta \mathcal{J}} \mathcal{W} = \int d^D x \delta_\sigma \mathcal{J} \frac{\delta}{\delta \mathcal{J}} S^{(2)} \equiv \int d^D x \mathcal{A}_\sigma \quad (1.2.8)$$

²One of the reasons we are interested in Weyl symmetry is because a theory in curved space which is invariant under Weyl transformations is automatically conformally invariant in flat space.

where we have parametrized the non-invariance of $S^{(2)}$ by $\mathcal{A}_\sigma$. The relation to the standard form in which the anomaly is presented will become clearer later on. This way of looking at things already clarifies some interesting point. First, the locality of $S^{(2)}$ directly implies that $\mathcal{A}_\sigma$ must also be a local function of the sources. Furthermore, since the variation of $S^{(2)}$ is equal to that of the renormalized effective action, it must forcefully be finite despite the fact that it is a divergent series of counterterms.

1.2.1.2 Weyl symmetry, the local RG equation and anomalous Ward identities

We now introduce a notation which is going to be prevalent throughout the rest of the thesis. As we have stated multiple times, the background fields transform under Weyl symmetry to compensate for its explicit breaking. Therefore the Weyl symmetry generator will be the combination of the variations of the individual sources:

$$\int d^2x \delta_\sigma \mathcal{J} \frac{\delta}{\delta \mathcal{J}} \equiv \Delta_\sigma = \Delta_\sigma^g - \Delta_\sigma^\beta \quad (1.2.9)$$

where

$$\Delta_\sigma^g = \int d^2x 2\sigma(x) g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} \quad (1.2.10)$$

$$\Delta_\sigma^\beta = - \int d^2x \delta_\sigma \lambda(x) \cdot \frac{\delta}{\delta \lambda(x)} \quad (1.2.11)$$

The Weyl variation is then taken to be of the most general respecting both dimensional analysis and diffeomorphism invariance, which yields:

$$\delta_\sigma \lambda^I(x) = -\sigma(x) \beta^I(\lambda(x)) \quad (1.2.12)$$

We also have that by dimensional analysis the various coefficients have to be a function of the couplings λ^I . We therefore finally have the following form for the generator of Weyl symmetry:

$$\Delta_\sigma = \int d^2x \left(2\sigma(x) g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} - \sigma(x) \beta^I(\lambda(x)) \frac{\delta}{\delta \lambda^I(x)} \right) \quad (1.2.13)$$

Now, as has already been stated in the previous section, the effective action is invariant up to the anomaly. Rewriting this using the new notation yields the local Callan-Symanzik equation:

$$\Delta_\sigma \mathcal{W} = \int d^2x \mathcal{A}_\sigma \quad (1.2.14)$$

To better understand the above equation, let us relate this way of treating symmetries to the more standard and better known Ward identities. We therefore do the following:

$$\Delta_\sigma \frac{\delta}{\delta \lambda^I(x)} \mathcal{W} = \left[\Delta_\sigma, \frac{\delta}{\delta \lambda^I(x)} \right] \mathcal{W} + \frac{\delta}{\delta \lambda^I(x)} \int d^2y \mathcal{A}_\sigma \quad (1.2.15)$$

where we have already used (1.2.14). Let us compute every term separately. We have:

$$\Delta_\sigma \frac{\delta}{\delta \lambda^I(x)} \mathcal{W} = \int d^2y \left(2\sigma(y) g^{\mu\nu}(y) \frac{\delta}{\delta g^{\mu\nu}(y)} \frac{\delta}{\delta \lambda^I(x)} - \sigma(y) \beta^J(\lambda(y)) \frac{\delta}{\delta \lambda^J(y)} \frac{\delta}{\delta \lambda^I(x)} \right) \mathcal{W} \quad (1.2.16)$$

We see from the form of Δ_σ that the only term in the commutator which is not going to cancel is the one where $\frac{\delta}{\delta\lambda^I(x)}$ acts on $\beta^J(\lambda(y))$. We therefore obtain:

$$\begin{aligned} \left[\Delta_\sigma, \frac{\delta}{\delta\lambda^I(x)} \right] \mathcal{W} &= \int d^2y \sigma(y) \frac{\delta}{\delta\lambda^I(x)} \left(\beta^J(\lambda(y)) \right) \frac{\delta}{\delta\lambda^J(x)} \mathcal{W} \\ &= \int d^2y \sigma(y) \partial_I \beta^J(\lambda(y)) \delta(x-y) \frac{\delta}{\delta\lambda^J(x)} \mathcal{W} \end{aligned} \quad (1.2.17)$$

So that finally, we have:

$$\begin{aligned} \int d^2y \sigma(y) \left[2g^{\mu\nu}(y) \frac{\delta}{\delta g^{\mu\nu}(y)} \frac{\delta}{\delta\lambda^I(x)} - \beta^J(\lambda(y)) \frac{\delta}{\delta\lambda^J(y)} \frac{\delta}{\delta\lambda^I(x)} \right] \mathcal{W} \\ = \int d^2y \sigma(y) \left[\partial_I \beta^J(\lambda(y)) \delta(x-y) \frac{\delta}{\delta\lambda^J(x)} \mathcal{W} + \frac{\delta}{\delta\lambda^I(x)} \mathcal{A} \right] \end{aligned} \quad (1.2.18)$$

where $\mathcal{A}$ is the anomaly where σ has been factored out.

Translated into the standard Ward identity language, the previous equation is equivalent to:

$$i\langle T(y) \mathcal{O}_I(x) \rangle - i\beta^J(\lambda(y)) \langle \mathcal{O}_J(y) \mathcal{O}_I(x) \rangle = \partial_I \beta^J(\lambda(y)) \delta(x-y) \langle \mathcal{O}_J(x) \rangle + \frac{\delta}{\delta\lambda^I(x)} \mathcal{A}[\mathcal{J}(y)] \quad (1.2.19)$$

In general the one-point function is zero (or the operator can be redefined such that this is true), but even if not we see that it only contributes when x and y coincide. One already sees the structure. If we were to have more insertions of $\mathcal{O}$ the left hand side would stay the same (up to factors of i) and essentially comes down to replacing T by $\beta^J \mathcal{O}_J$. So what we are doing is expanding T in a basis of scalar operators. It turns out that there are some subtleties which we have glossed over in this simplified treatment but the basic idea is there. The right hand side of the equation is a collection of contact terms which contribute only when two or more point coincide and their coefficient are proportional to the variations of the coefficients in the Weyl generators. The variation of the anomaly finally yields ultra-local terms for which every point coincides as we will confirm later on.

* * * * *

To illustrate what we have said above more concretely, let us take a look at what happens for 3 insertions of $\mathcal{O}$. The computation of the commutator yields:

$$\begin{aligned} \left[\Delta_\sigma, \frac{\delta}{\delta\lambda^{I_1}(x_1)} \frac{\delta}{\delta\lambda^{I_2}(x_2)} \frac{\delta}{\delta\lambda^{I_3}(x_3)} \right] \mathcal{W} &= \int d^2y \left[-\beta^J \delta_J \delta_{I_1} \delta_{I_2} \delta_{I_3} \mathcal{W} + \delta_{I_1} \delta_{I_2} \delta_{I_3} (\beta^J \delta_J \mathcal{W}) \right] \\ &= \int d^2y \left[\partial_{I_1} \partial_{I_2} \partial_{I_3} \beta^J \delta(x_2-y) \delta(x_1-y) \delta(x_3-y) \delta_J \mathcal{W} \right. \\ &\quad + (\partial_{I_2} \partial_{I_3} \beta^J \delta(x_2-y) \delta(x_3-y) \delta_J \delta_{I_1} \mathcal{W} + \partial_{I_1} \partial_{I_3} \beta^J \delta(x_1-y) \delta(x_3-y) \delta_J \delta_{I_2} \mathcal{W} \\ &\quad \left. + \partial_{I_2} \partial_{I_1} \beta^J \delta(x_2-y) \delta(x_1-y) \delta_J \delta_{I_3} \mathcal{W}) \right. \\ &\quad \left. + (\partial_{I_1} \beta^J \delta(x_1-y) \delta_{I_2} \delta_{I_3} \delta_J \mathcal{W} + \partial_{I_2} \beta^J \delta(x_2-y) \delta_{I_1} \delta_{I_3} \delta_J \mathcal{W} + \partial_{I_3} \beta^J \delta(x_3-y) \delta_{I_2} \delta_{I_1} \delta_J \mathcal{W}) \right] \end{aligned} \quad (1.2.20)$$

So that finally the Ward identity is:

$$-i \left(\langle T(y) \mathcal{O}_{I_1}(x_1) \mathcal{O}_{I_2}(x_2) \mathcal{O}_{I_3}(x_3) \rangle - \beta^J(\lambda(y)) \langle \mathcal{O}_J(y) \mathcal{O}_{I_1}(x_1) \mathcal{O}_{I_2}(x_2) \mathcal{O}_{I_3}(x_3) \rangle \right) \quad (1.2.21)$$

$$\begin{aligned} &= \delta(x_2 - y) \delta(x_1 - y) \delta(x_3 - y) \partial_{I_1} \partial_{I_2} \partial_{I_3} \beta^J \langle \mathcal{O}_J(y) \rangle \\ &\quad + i(\delta(x_2 - y) \delta(x_3 - y) \partial_{I_2} \partial_{I_3} \beta^J \langle \mathcal{O}_{I_1}(x_1) \mathcal{O}_J(y) \rangle + \delta(x_1 - y) \delta(x_3 - y) \partial_{I_1} \partial_{I_3} \beta^J \langle \mathcal{O}_{I_2}(x_2) \mathcal{O}_J(y) \rangle \\ &\quad + \delta(x_2 - y) \delta(x_1 - y) \partial_{I_2} \partial_{I_1} \beta^J \langle \mathcal{O}_{I_3}(x_3) \mathcal{O}_J(y) \rangle) \\ &\quad + (-1)(\delta(x_1 - y) \partial_{I_1} \beta^J \langle \mathcal{O}_{I_3}(x_3) \mathcal{O}_{I_2}(x_2) \mathcal{O}_J(y) \rangle + \delta(x_2 - y) \partial_{I_2} \beta^J \langle \mathcal{O}_{I_3}(x_3) \mathcal{O}_{I_1}(x_1) \mathcal{O}_J(y) \rangle \\ &\quad + \delta(x_3 - y) \partial_{I_3} \beta^J \langle \mathcal{O}_{I_1}(x_1) \mathcal{O}_{I_2}(x_2) \mathcal{O}_J(y) \rangle) \\ &\quad + \frac{\delta}{\delta \lambda^{I_1}(x_1)} \frac{\delta}{\delta \lambda^{I_2}(x_2)} \frac{\delta}{\delta \lambda^{I_3}(x_3)} \mathcal{A}[\mathcal{J}(y)] \end{aligned} \quad (1.2.22)$$

which confirms the structure we mentioned in the previous paragraph, namely that on the left hand side we simply replace T by $\beta^I \mathcal{O}_I$ and on the right hand side we have contact terms of all types where two, three or four points coincide and whose coefficient is given by a derivative of the β function.

Another example is what happens for the two-point function of T .

$$\begin{aligned} \int d^2x d^2y \sqrt{-g(x)} \sqrt{-g(y)} \sigma_1(x) \sigma_2(y) \langle T(x) T(y) \rangle &= (-i) \Delta_{\sigma_1}^g \Delta_{\sigma_2}^g \mathcal{W} \\ &= (-i) \left(\Delta_{\sigma_1}^g \Delta_{\sigma_2}^W + \Delta_{\sigma_1}^g \Delta_{\sigma_2}^\beta \right) \mathcal{W} \\ &= (-i) \left(\Delta_{\sigma_1}^g \int d^2y \mathcal{A}_\sigma^W[g, \lambda] + \Delta_{\sigma_2}^\beta \Delta_{\sigma_1}^g \mathcal{W} + [\Delta_{\sigma_1}^g, \Delta_{\sigma_2}^\beta] \mathcal{W} \right) \\ &= (-i) \left(\Delta_{\sigma_1}^g \int d^2y \mathcal{A}_{\sigma_2}^W[g, \lambda] + \Delta_{\sigma_2}^\beta \int d^2x \mathcal{A}_{\sigma_1}^W[g, \lambda] + \Delta_{\sigma_2}^\beta \Delta_{\sigma_1}^\beta \mathcal{W} + [\Delta_{\sigma_1}^g, \Delta_{\sigma_2}^\beta] \mathcal{W} \right) \end{aligned}$$

The commutator is clearly zero. We thus continue:

$$\begin{aligned} &= (-i) \int d^2x d^2y \left(2\sigma_1(x) g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} \mathcal{A}_{\sigma_2}^W + \sigma_2(y) \beta^I \frac{\delta}{\delta \lambda^I(y)} \mathcal{A}_{\sigma_1}^W + \sigma_2(y) \beta^J \frac{\delta}{\delta \lambda^J(y)} \sigma_1(x) \beta^I \frac{\delta}{\delta \lambda^I(x)} \mathcal{W} \right) \\ &= (-i) \int d^2x d^2y \sigma_1(x) \sigma_2(y) \left(2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} \frac{\mathcal{A}_{\sigma_2}^W}{\sigma_2} + \beta^I \frac{\delta}{\delta \lambda^I(y)} \frac{\mathcal{A}_{\sigma_1}^W}{\sigma_1} + \right. \\ &\quad \left. \beta^J(\lambda(y)) \frac{\partial \beta^I(\lambda(x))}{\partial \lambda^J} \delta(x - y) \langle \mathcal{O}_I(x) \rangle + \beta^I(\lambda(x)) \beta^J(\lambda(y)) i \langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle \right) \end{aligned}$$

Finally we have obtained:

$$\begin{aligned} \langle T(x) T(y) \rangle &= (-i) 2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} \frac{\mathcal{A}_{\sigma_2}^W[g, \lambda(y)]}{\sigma_2} \Big|_{g=\eta} + (-i) \beta^I \frac{\delta}{\delta \lambda^I(y)} \frac{\mathcal{A}_{\sigma_1}^W[\eta, \lambda(x)]}{\sigma_1} + \\ &\quad (-i) \beta^J(\lambda(y)) \frac{\partial \beta^I(\lambda(x))}{\partial \lambda^J} \delta(x - y) \langle \mathcal{O}_I(x) \rangle + \beta^I(\lambda(x)) \beta^J(\lambda(y)) \langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle \end{aligned} \quad (1.2.23)$$

We see that except for the last term, they are all local terms only appearing at $x = y$. We emphasize that the notation we used for the terms related to the anomaly is only symbolic, so as to be able to write a standard Ward identity. Indeed, the anomalies can depend on derivatives of σ , in which case one needs to perform some integration by parts.

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There is one more crucial thing which needs to be mentioned. The Δ_σ are generators of the Weyl symmetry. They therefore have to satisfy commutation relations of the Lie algebra of Weyl symmetry, which actually is an Abelian Lie algebra, which means that they have to satisfy:

$$[\Delta_{\sigma_1}, \Delta_{\sigma_2}] = 0 \quad (1.2.24)$$

In the simplified case we are currently discussing, these commutation relations are trivially satisfied, but it turns out that in a more complete treatment, this constraint gives relations between the various coefficients of the Weyl generator.

1.2.1.3 The structure of the Weyl anomaly and the consistency conditions

1.2.1.3.1 The conformal anomaly We have seen previously that:

$$\Delta_\sigma \mathcal{W} = \int d^2x \mathcal{A}_\sigma[\mathcal{J}] \quad (1.2.25)$$

and identified $\mathcal{A}_\sigma$ as the term which gives rise to anomalies. Before treating the general case, let us start by reviewing the anomaly which arises when one puts a CFT in a curved background, called alternatively the Weyl, Conformal or Trace anomaly. Notice that in our formalism, this means taking $\lambda^I = 0$.

The way this anomaly was first discovered is through explicit calculations, which are very cumbersome (see [10] and references therein). We are going to approach this subject in a slightly different way, by assuming the existence of anomalies and deducing their form by imposing a set of reasonable conditions. This way of approaching things is at the basis of the whole local RG formalism presented in this thesis. Indeed, the Weyl coefficients presented in the previous section were also chosen to have the most general form compatible with a set of conditions. Of course, it could happen that not all the anomalies compatible with our conditions actually arise, but it turns out that this is not the case (following the saying the in quantum field theory “Everything that is possible is compulsory”).

The question now becomes: What is the most general form $\mathcal{A}_\sigma$ can take? To answer this question, let us see what conditions we impose on $\mathcal{A}_\sigma$:

1. We take $\mathcal{A}_\sigma$ to be a local functional of the metric. We have already justified this at the end of section 1.2.1.1. The argument essentially is that $\mathcal{A}_\sigma$ corresponds to the variation of the part of the action which correspond to counterterms, which are implemented by a local functional of the sources.
2. $\mathcal{A}_\sigma$ has to be diffeomorphism invariant. This is intuitively evident, but can also be verified by checking that $[\Delta^{\text{diff}}, \Delta_\sigma] = 0$.
3. $\mathcal{A}_\sigma$ has to be linear in σ since it is an infinitesimal variation.
4. $\mathcal{A}_\sigma$ has to be of dimension D , i.e. it has to satisfy dimensional analysis.

There is one more condition which we will impose but it stands somewhat apart from the other four. In the previous section, we already mentioned that since Δ_σ s are generators of the Weyl symmetry they have to satisfy the commutation relations of the corresponding

Lie algebra. When supplemented with the local Callan-Symanzik equation, this implies that $\mathcal{A}_\sigma$ has to satisfy the Wess-Zumino (WZ) consistency conditions, which is a sort of integrability condition which constrains the possible form the anomalies can take. They simply arise from applying the vanishing commutator of the generators to the generating functional:

$$[\Delta_{\sigma_1}, \Delta_{\sigma_2}] \mathcal{W}[\mathcal{J}] = 0 \quad \Leftrightarrow \quad \Delta_{\sigma_1} \int d^D x \mathcal{A}_{\sigma_2}[\mathcal{J}] = \Delta_{\sigma_2} \int d^D x \mathcal{A}_{\sigma_1}[\mathcal{J}] \quad (1.2.26)$$

The terms which satisfy the conditions are:

- $\sqrt{-g(x)}\sigma(x)R$
- $\sqrt{-g(x)}\square\sigma$
- $\sqrt{-g(x)}\sigma(x)\Lambda^2$

Before we proceed, we need to mention a crucial aspect of anomalies. As we have already argued previously, anomalies are directly related to the renormalization procedure and as such they possess the same arbitrariness inherent in the renormalization procedure. In the formalism we use in this thesis this arbitrariness corresponds to the freedom of adding local functionals of the sources to the generating functional. It is therefore very important to realize that anomalies are a scheme dependent notion.

Let us now proceed with our analysis. The second term is of no interest since it forms a total derivative. We then can write:

$$\Delta_\sigma \mathcal{W}[\mathcal{J}] = \int d^2 x \sqrt{-g(x)} \kappa_1 \sigma(x) R(x) \quad (1.2.27)$$

Under Weyl transformations, the Ricci scalar transforms as:

$$\begin{aligned} \tilde{R} &= e^{2\sigma} \left(R + 2(\square\sigma - (\partial\sigma)^2) + 2(\partial\sigma)^2 \right) \\ &= e^{2\sigma} (R + 2\square\sigma) \\ &\approx R + 2\square\sigma + 2R\sigma \end{aligned} \quad (1.2.28)$$

Furthermore, we also have:

$$\widetilde{\sqrt{-g}} \approx -D\sigma\sqrt{-g} \quad (1.2.29)$$

Using this we obtain:

$$\begin{aligned} \Delta_{\sigma_2} \int d^2 x \sqrt{-g(x)} \sigma_1(x) \kappa_1 R &= \kappa_1 \int d^2 x \left[-2\sigma_2 \sqrt{-g(x)} \sigma_1 R + \sqrt{-g(x)} \sigma_1 (2\square\sigma_2 + 2R\sigma_2) \right] \\ &= 2\kappa_1 \int d^2 x \sqrt{-g(x)} [\sigma_1 \sigma_2 R - \sigma_1 \sigma_2 R + \sigma_1 \square\sigma_2] \\ &= 2\kappa_1 \int d^2 x \sqrt{-g(x)} \sigma_1 \square\sigma_2 \end{aligned} \quad (1.2.30)$$

So that finally we have obtained:

$$[\Delta_{\sigma_1}, \Delta_{\sigma_2}] \mathcal{W}[\mathcal{J}] = \int d^2 x \sqrt{-g(x)} 2\kappa_1 (\sigma_2 \square\sigma_1 - \sigma_1 \square\sigma_2) = 0 \quad (1.2.31)$$

where in the last step we have used an integration by parts. Notice that this depends crucially on the fact that κ_1 is a constant and not a function of the λ^I s as it will be once we move off criticality.

The last “cosmological constant” anomaly trivially satisfies the WZ consistency conditions.

We therefore conclude that the trace anomaly in two dimensions is given by:

$$\mathcal{A}_\sigma[g] = \kappa_1 \sqrt{-g(x)} \sigma(x) R + f_2 \sqrt{-g(x)} \sigma(x) \Lambda^2 \quad (1.2.32)$$

To see why this is called the trace anomaly, we can use the formula (1.2.25) and write:

$$\langle T(x) \rangle = \kappa_1 R(x) + f_2 \Lambda^2 \quad (1.2.33)$$

This explicitly shows that $\mathcal{A}_\sigma$ indeed generates the standard anomaly.

1.2.1.3.2 The Weyl anomaly off-criticality Let us now move off-criticality. The terms compatible with the conditions then are:

- $\sqrt{-g} \sigma \left(\frac{1}{2} c R \right)$
- $\sqrt{-g} \sigma \left(-\frac{1}{2} \chi_{IJ} \nabla_\mu \lambda^I \nabla^\mu \lambda^J \right)$
- $\sqrt{-g} \nabla^\mu \sigma \left(-w_I \nabla_\mu \lambda^I \right)$
- $\sqrt{-g} \sigma \Lambda_2$

where c , χ_{IJ} and w_I are now functions of the λ_I .

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Let us now impose the WZ consistency conditions. We have:

$$\begin{aligned} \Delta_{\sigma_2}^g \int d^2x \mathcal{A}_{\sigma_1}[\mathcal{J}(y)] &= \int d^2x \sqrt{-g(x)} (c(\lambda(x)) \sigma_1(x) \square \sigma_2(x)) + (\dots) \\ &= \int d^2x \sqrt{-g(x)} [-\nabla^\mu c(\lambda(x)) \sigma_1(x) \nabla_\mu \sigma_2(x) + (\dots)] \end{aligned} \quad (1.2.34)$$

Some comments are in order: In the first line we used the result of (1.2.30) by simply replacing κ_1 by $\frac{1}{2}c$. Furthermore, the $\dots$ represent terms which are proportional to $\sigma_1 \sigma_2$ coming from the variations of $\sqrt{-g} g^{\mu\nu}$ in the two other terms of the anomaly which trivially vanish in the commutator. In the second line, we have simply integrated by parts and absorbed one of the terms of the product rule which is symmetric under $\sigma_1 \leftrightarrow \sigma_2$ into the $\dots$.

The second term yields:

$$\begin{aligned}
\Delta_{\sigma_2}^\beta \int d^2x \mathcal{A}_{\sigma_1}[\mathcal{J}(y)] &= - \int d^2y d^2x \sqrt{-g(x)} \sigma_2(y) \beta^K(\lambda(y)) \\
&\quad \times \left[\frac{1}{2} \chi_{KJ} \nabla_\mu^x \delta(x-y) \nabla^\mu \lambda^J(x) \sigma_1(x) + \frac{1}{2} \chi_{IK} \nabla_\mu \lambda^I(x) \nabla^{x,\mu} \delta(x-y) \sigma_1(x) \right. \\
&\quad \left. + \nabla^\mu \sigma_1(x) \partial_K w_I \delta(x-y) \nabla_\mu \lambda^I(x) + \nabla^\mu \sigma_1(x) w_K \nabla_\mu^x \delta(x-y) + (\dots) \right] \\
&= - \int d^2y d^2x \sqrt{-g(x)} \sigma_2(y) \beta^K(\lambda(y)) \left[\chi_{KI} \left(-\nabla_\mu^y \delta(x-y) \right) \nabla^\mu \lambda^I(x) \sigma_1(x) \right. \\
&\quad \left. + \nabla^\mu \sigma_1(x) \partial_K w_I \delta(x-y) \nabla_\mu \lambda^I(x) + \nabla^\mu \sigma_1(x) w_K \left(-\nabla_\mu^y \delta(x-y) \right) + (\dots) \right] \\
&= - \int d^2x \sqrt{-g(x)} \left(\sigma_1(x) \nabla_\mu \sigma_2(x) \left[\beta^K(\lambda(x)) \chi_{KI} \nabla^\mu \lambda^I(x) \right] \right. \\
&\quad \left. + \sigma_2(x) \nabla_\mu \sigma_1(x) \left[\beta^K(\lambda(x)) \partial_K w_I \nabla^\mu \lambda^I + \nabla^\mu \beta^K(\lambda(x)) w_K \right] + (\dots) \right) \tag{1.2.35}
\end{aligned}$$

Combining the two yields:

$$0 = [\Delta_{\sigma_1}, \Delta_{\sigma_2}] \mathcal{W}[\mathcal{J}] = \int d^2x \sqrt{-g(x)} (\sigma_2(x) \nabla_\mu \sigma_1(x) - \sigma_1(x) \nabla_\mu \sigma_2(x)) V^\mu \tag{1.2.36}$$

where V^μ is given by:

$$\begin{aligned}
V^\mu &= \nabla^\mu c(\lambda(x)) + \left[\beta^K(\lambda(x)) \partial_K w_I \nabla^\mu \lambda^I + \nabla^\mu \beta^K(\lambda(x)) w_K \right] - \beta^K(\lambda(x)) \chi_{KI} \nabla^\mu \lambda^I(x) \\
&= \nabla^\mu \lambda^I \left\{ \partial_I c(\lambda(x)) + \left[\beta^K(\lambda(x)) \partial_K w_I + \partial_I \beta^K(\lambda(x)) w_K \right] - \beta^K(\lambda(x)) \chi_{KI} \right\} \tag{1.2.37}
\end{aligned}$$

Since the λ_I s are arbitrary, we have obtained the constraint:

$$\partial_I c(\lambda) = \chi_{IK} \beta^K(\lambda) - \left[\beta^K(\lambda) \partial_K w_I + \partial_I \beta^K(\lambda) w_K \right] \tag{1.2.38}$$

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Imposing the consistency conditions yields:

$$\partial_I c(\lambda) = \chi_{IK} \beta^K(\lambda) - \mathcal{L}_\beta[w_I] \tag{1.2.39}$$

where we have defined the Lie derivative along β by

$$\mathcal{L}_\beta[w_I] = \beta^K(\lambda) \partial_K w_I + \partial_I \beta^K(\lambda) w_K \tag{1.2.40}$$

We now define

$$\tilde{c} = c + w_I \beta^I \tag{1.2.41}$$

Notice that at a conformal fixed point $\tilde{c}$ reduces to the standard trace anomaly (up to maybe a factor). The new function $\tilde{c}$ then obeys the following relation:

$$\partial_I \tilde{c}(\lambda) = (\chi_{IK} + [\partial_I w_K - \partial_K w_I]) \beta^K(\lambda) \tag{1.2.42}$$

If we then contract, the above relation with β^I , we finally obtain:

$$\mu \frac{d}{d\mu} \tilde{c} \equiv \beta^I \partial_I \tilde{c} = \chi_{IK} \beta^I \beta^K \tag{1.2.43}$$

We therefore have a perturbative proof for the gradient flow formula (up to the additional antisymmetric “matrix” which vanishes upon contraction with β^J). That being said, this result is only interesting if we that the matrix χ_{IJ} is positive definite, which is part of what follows.

* * * * *

In this separated section, we present a quick way of proving that χ_{IJ} is positive to leading order. We work in Euclidean space. At the conformal fixed point, the two-point function of the scalar operators is given by (see appendix A.2):

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle = \frac{\delta_{IJ}}{x^4} \quad (1.2.44)$$

where we have scaled the operators such that the numerical coefficient which generally appears next to the identity matrix is unity. Notice that this form is a consequence of unitarity, which is a prerequisite for the proof.

This expression has an IR divergence when $x \rightarrow 0$, which needs to be regulated, which will generate a dependence on the renormalization scale μ . We will use the method of differential regularization to regularize this expression. For details, see [11]. The idea is to soften the divergence by rewriting the expression as the derivative of a less divergent quantity. In our case, we have:

$$\frac{1}{x^4} = \square^2 H(x^2) \quad (1.2.45)$$

where we require H to be a function of x^2 by rotational invariance. This gives the differential equation:

$$\frac{1}{r} \partial_r \left(r \partial_r \left(\frac{1}{r} \partial_r (ru') \right) \right) = \frac{1}{r^4} \quad (1.2.46)$$

whose solution is given by:

$$H(x) = \frac{1}{8} \ln^2 x\mu + (2\pi)c \ln x\mu \quad (1.2.47)$$

where we have dropped some terms which do not play any role in the regularization. Notice the appearance of the renormalization scale μ . We then have the following regularized expression for the two-point function:

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle = \delta_{IJ} \left(\square^2 \frac{1}{8} \ln^2 x\mu + c \square \delta(x) \right) \quad (1.2.48)$$

where we have already used that in two dimensions, we have:

$$\square \ln(x) = (2\pi) \delta(x) \quad (1.2.49)$$

as can be easily checked by writing $\ln(x + \epsilon)$ and integrating over the resulting expression. The point of this is that now that we have the dependence on μ , we can look at the RG behavior of the two-point function:

$$\mu \frac{\partial}{\partial \mu} \langle \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle = \delta_{IJ} \square^2 \frac{1}{4} \ln x\mu = \frac{\delta_{IJ}}{4} \square \delta(x) \quad (1.2.50)$$

The way to relate this to χ_{IJ} is to notice that the RG evolution of the scalar two-point function is related to correlators of T . Indeed, using equation (1.2.56)

$$\Delta^\mu \mathcal{W} \equiv \left[\mu \frac{\partial}{\partial \mu} + \int d^2x 2g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} \right] \mathcal{W} = 0 \quad (1.2.51)$$

we obtain:

$$\Delta^\mu \frac{\delta}{\delta \lambda^I(x)} \frac{\delta}{\delta \lambda^J(0)} \mathcal{W} = 0 \quad \Rightarrow \quad \int d^2y \langle T(y) \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle = \mu \frac{\partial}{\partial \mu} \langle \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle \quad (1.2.52)$$

We can now find an alternative expression for the correlator of T in the following way:

$$\Delta_\sigma \frac{\delta}{\delta \lambda^I(x)} \frac{\delta}{\delta \lambda^J(0)} \mathcal{W} = \left[\Delta_\sigma, \frac{\delta}{\delta \lambda^I(x)} \frac{\delta}{\delta \lambda^J(0)} \right] \mathcal{W} + \frac{\delta}{\delta \lambda^I(x)} \frac{\delta}{\delta \lambda^J(0)} \int d^2y \mathcal{A}_\sigma \quad (1.2.53)$$

We can evaluate this expression at the conformal fixed point since we are interested in the leading order result, which then yields:

$$\int d^2y \langle T(y) \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle = \chi_{IJ} \square \delta(y) \quad (1.2.54)$$

We can therefore now do the identification which yields:

$$\chi_{IJ} = \frac{\delta_{IJ}}{4} \quad (1.2.55)$$

which proves that χ_{IJ} is positive definite at leading order. This result is robust in the sense that the leading order is essentially unity and any correction will be of order λ , which in perturbation theory, when λ is small, will not change the positive definiteness of χ_{IJ} .

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1.2.1.4 Global Callan-Symanzik equation and the generators scale and conformal symmetry

To understand Weyl symmetry, let us relate it to other types of scale transformations. The only dimensionful parameter in $\mathcal{W}$ is the renormalization scale. We therefore have that:

$$\Delta^\mu \mathcal{W} \equiv \left[\mu \frac{\partial}{\partial \mu} + \int d^2x 2g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} \right] \mathcal{W} = 0 \quad (1.2.56)$$

which simply states that the change of scale can be compensated by a change of metric. We can now define a generator in which the metric is absent by doing the following:

$$\Delta^{RG} \equiv \Delta^\mu + \Delta_{\sigma=-1} = \mu \frac{\partial}{\partial \mu} + \int d^2x \beta^I \frac{\delta}{\delta \lambda^I(x)} \quad (1.2.57)$$

We see that this is essentially the ordinary Callan-Symanzik equation operator applied to the case of local sources. This expression shows a very important fact:

$$\Delta^{RG} \mathcal{W} = \int d^2x \mathcal{A}_{-1} \quad (1.2.58)$$

which establishes a connection between the anomaly and the renormalization scale dependence of $\mathcal{W}$, i.e. the dependence on $\ln \mu$.

Another generator of interest is the generator of diffeomorphisms, which is trivially given by:

$$\Delta_\xi^{\text{diff}} = \int d^2x \left[(\nabla_\rho \xi^\mu g^{\rho\nu} + \nabla_\rho \xi^\nu g^{\mu\rho}) \frac{\delta}{\delta g^{\mu\nu}} - \xi^\rho \nabla_\rho \lambda^I \frac{\delta}{\delta \lambda^I} \right] \quad (1.2.59)$$

This then directly yields:

$$\Delta_c^D \equiv \Delta_{cx}^{\text{diff}} + \Delta_{-c} = c \int d^2x \left[\beta^I \frac{\delta}{\delta \lambda^I} - x^\rho \nabla_\rho \lambda^I \frac{\delta}{\delta \lambda^I} \right] \quad (1.2.60)$$

$$\begin{aligned} \Delta_b^K &\equiv \Delta_{2(b \cdot x)x^\mu - x^2 b^\mu}^{\text{diff}} + \Delta_{-2b \cdot x} \\ &= \int d^2x \left[x 2(b \cdot x) \beta^I \frac{\delta}{\delta \lambda^I} - (2(b \cdot x)x^\mu - x^2 b^\mu) \nabla_\mu \lambda^I \frac{\delta}{\delta \lambda^I} \right] \end{aligned} \quad (1.2.61)$$

In the simplified case we are considering, these relations do not tell us anything surprising. Indeed, they simply imply that in the “zero” background, a theory is scale and conformal invariant if $\beta = 0$. Notice that this does not imply that scale and conformal invariance are equivalent in two dimensions. Indeed, we remind the reader that we are considering a simplified theory with only marginal deformations. In a more complete analysis, there are indeed some subtleties which arise, some of which we will discuss in the four-dimensional case. It actually does turn out to be true (up to some technical assumption) that scale invariance implies conformal invariance in two dimensions and this result is called Polchinski’s theorem[12].

1.2.1.5 The dilaton and its effective action

In this section, we will introduce the dilaton and its effective action to rewrite the correlators of T in terms of the other composite operators and local terms associated with the variation of the anomaly. We have already done this in a very basic form in section 1.2.1.2. This section follows the sections 3.1 to 3.3 of [7] very closely. Most of our notation is also taken from that reference.

So let us now introduce the dilaton into the mix. The most convenient way of doing so is by introducing a “new” metric:

$$\hat{g}^{\mu\nu}(x) = e^{2\tau(x)} g^{\mu\nu}(x) \quad (1.2.62)$$

where τ is now a dilaton background field. We then define the dilaton effective action $\Gamma[g, \tau]$ to be the effective action $\mathcal{W}$ evaluated in the dilaton background:

$$\mathcal{J}_1(g, \tau) \equiv \left(\hat{g}_{\mu\nu}(x) = e^{-2\tau(x)} g_{\mu\nu}(x), \lambda^I = \lambda^I(\mu) = \text{const} \right) \quad (1.2.63)$$

To get a better feeling of what this dilaton effective action represents, let us do an expansion:

$$\Gamma[g, \tau] = \mathcal{W}[\mathcal{J}_1] = \exp[\Delta_\tau^g] \mathcal{W}[\mathcal{J}] \Big|_{\mathcal{J}=\mathcal{J}_0} = \sum_{n=0}^{\infty} \frac{1}{n!} (\Delta_\tau^g)^n \mathcal{W}[\mathcal{J}] \Big|_{\mathcal{J}=\mathcal{J}_0} \quad (1.2.64)$$

where we have used the fact that Δ_τ^g is the generator of Weyl symmetry for the metric and therefore exponentiated gives the finite transformation of the metric starting from the background $\mathcal{J}_0$ which we have defined as:

$$\mathcal{J}_0(g) \equiv \left(\hat{g}^{\mu\nu} = g^{\mu\nu}, \lambda^I = \lambda^I(\mu) = \text{const} \right) \quad (1.2.65)$$

Then simply using our definition (1.2.4) of the correlators of T , we obtain:

$$\Gamma[\eta, \tau] = \sum_{n=0}^{\infty} \frac{i^{n-1}}{n!} \int d^2x_n \cdots \int d^2x_1 \tau(x_n) \cdots \tau(x_1) \langle T(x_1) \cdots T(x_n) \rangle \quad (1.2.66)$$

We therefore have first reason for why the dilaton is useful: It packages the correlators of T . Another way to see the same thing is:

$$\frac{\delta}{\delta\tau(x)} \mathcal{W}[g, \tau] = \int d^4y \frac{\delta\hat{g}^{\mu\nu}(y)}{\delta\tau(x)} \frac{\delta\mathcal{W}[\hat{g}]}{\delta\hat{g}^{\mu\nu}(y)} = 2\hat{g}^{\mu\nu}(x) \frac{\delta\mathcal{W}[\hat{g}]}{\delta\hat{g}^{\mu\nu}(x)} = \sqrt{-\hat{g}(x)} \langle T(x) \rangle \quad (1.2.67)$$

Notice one important fact: By introducing the dilaton, we created a redundancy akin to gauge redundancy. Indeed, the following transformation

$$\tau \rightarrow \tau - \alpha \quad \& \quad g^{\mu\nu} \rightarrow e^{2\alpha} g^{\mu\nu} \quad (1.2.68)$$

which we are going to call “gauge transformation” leaves $\hat{g}$ unaffected.

One can now notice another advantage of introducing the dilaton. Indeed, Weyl transformations are now simply implemented as a shift symmetry of the dilaton:

$$\tau \rightarrow \tau + \sigma \quad \& \quad g_{\mu\nu} \rightarrow g_{\mu\nu} \quad (1.2.69)$$

Recall that our goal was to find a way to relate the correlators of T to correlators of the other composite operators. To do so, we need to shift the dilaton dependence into λ^I by considering the Weyl transformed sources:

$$\mathcal{J}_2(g, \tau) \equiv \exp(-\Delta_\tau) \mathcal{J}|_{\mathcal{J}=\mathcal{J}_1} = (g^{\mu\nu}, \lambda^I[\tau]) \quad (1.2.70)$$

We can then separate the dilaton effective action into two parts:

$$\Gamma[g, \tau] = (\mathcal{W}[\mathcal{J}_1] - \mathcal{W}[\mathcal{J}_2]) + \mathcal{W}[\mathcal{J}_2] \equiv \Gamma_{\text{local}}[g, \tau] + \Gamma_{\text{non-local}}[g, \tau] \quad (1.2.71)$$

Γ_{local} is local and therefore deserves its name because it simply corresponds to a finite Weyl variation of $\mathcal{W}$, whereas in $\Gamma_{\text{non-local}}$, the $\lambda^I[\tau]$ act as sources for the $\mathcal{O}_I$.

To compute the local part of the dilaton effective action, we define a source which interpolates between the sources $\mathcal{J}_1$ and $\mathcal{J}_2$. This is useful because as we already mentioned Γ_{local} is a finite Weyl variation of $\mathcal{W}$ and by defining an interpolating source, it gives a way to relate that finite variation to the integral over infinitesimal variations along the interpolation. These infinitesimal variations are then related to the infinitesimal variations of $\mathcal{W}$ which in turn are related to the anomaly. Mathematically, this goes as follows. The interpolating source is defined as:

$$\mathcal{J}_{1+y}(g, \tau) \equiv \exp(-y\Delta_\tau) \mathcal{J}|_{\mathcal{J}=\mathcal{J}_1} = (g^{\mu\nu} e^{2(1-y)\tau}, \lambda^I[\tau, y]) \quad (1.2.72)$$

What we said in words earlier then translates to:

$$\Gamma_{\text{local}} = - \int_0^1 \frac{d}{dy} \mathcal{W}[\mathcal{J}_{1+y}] dy = - \int_0^1 (-\Delta_\tau \mathcal{W}[\mathcal{J}_{1+y}]) dy = \int d^2x \int_0^1 dy \mathcal{A}_\tau(\mathcal{J}_{1+y}) \quad (1.2.73)$$

Now let us start by computing explicitly $\mathcal{J}_{1+y}$. Simply by its definition, it satisfies the following set of first order differential equations:

$$\frac{d}{dy} \mathcal{J}_{1+y} = -\Delta_\tau \mathcal{J}_{1+y} \quad (1.2.74)$$

For the couplings this yields the equation:

$$\begin{aligned} \frac{d}{dy} \lambda^I[\tau, y] &= - \left(- \int d^2x \tau(x) \beta^J(\lambda[\tau, y]) \frac{\delta}{\delta \lambda^J[\tau, y](x)} \lambda^I[\tau, y] \right) \\ &= \tau \beta^I(\lambda[\tau, y]) \end{aligned} \quad (1.2.75)$$

which, supplemented with the initial condition $\lambda[\tau, 0] = \lambda(\mu)$, has the solution:

$$\lambda^I[\tau, y] = \lambda^I(\mu e^{y\tau}) \quad (1.2.76)$$

Indeed, we have:

$$\begin{aligned} \frac{d}{dy} \lambda^I(\mu e^{y\tau}) &= \mu \tau e^{y\tau} \frac{d}{d\bar{\mu}} \lambda^I(\bar{\mu}) \\ &= \tau \bar{\mu} \frac{d}{d\bar{\mu}} \lambda^I(\bar{\mu}) \\ &= \tau B^I(\lambda(\bar{\mu})) \\ &= \tau \beta^I(\lambda[\tau, y]) \end{aligned}$$

* * * * *

Let us now look separately at each term of the anomaly evaluated in the interpolated background. To do so, we will use the following results:

$$\sqrt{-\hat{g}[y]} = \sqrt{-g} e^{-2(1-y)\tau} \quad (1.2.77)$$

$$R(\hat{g}^{\mu\nu}[y]) = e^{2(1-y)\tau} \left(R(g^{\mu\nu}) + 2(1-y) \nabla^2 \tau \right) \quad (1.2.78)$$

$$\nabla_\mu \lambda^I(\mu e^{y\tau}) = y \nabla_\mu \tau \beta^I(\lambda^I(\mu e^{y\tau})) \quad (1.2.79)$$

$$\nabla_\mu f(\lambda^I(\mu e^{y\tau})) = y \nabla_\mu \tau \beta^I(\lambda^I(\mu e^{y\tau})) \partial_I f(\lambda^I(\mu e^{y\tau})) \quad (1.2.80)$$

$$\frac{d}{dy} f(\mu e^{y\tau}) = \tau \beta^I(\lambda^I(\mu e^{y\tau})) \partial_I f(\lambda^I(\mu e^{y\tau})) \quad (1.2.81)$$

The first “gravitational” term looks like:

$$\begin{aligned} & \int d^2x \int d^2y \sqrt{-\hat{g}[y](x)} \tau(x) \left(\frac{1}{2} c(\lambda^I(\mu e^{y\tau})) R(\hat{g}^{\mu\nu}[y](x)) \right) \\ &= \int d^2x \int d^2y \sqrt{-g} e^{-2(1-y)\tau} \tau(x) \left(\frac{1}{2} c(\lambda^I(\mu e^{y\tau})) e^{2(1-y)\tau} \left(R(g^{\mu\nu}) + 2(1-y) \nabla^2 \tau \right) \right) \\ &= \int d^2x \int d^2y \sqrt{-g} \left(\frac{1}{2} \tau(x) c(\lambda^I(\mu e^{y\tau})) R(g^{\mu\nu}) + c(\lambda^I(\mu e^{y\tau})) (1-y) \tau \nabla^2 \tau \right) \\ &= \int d^2x \sqrt{-g} \left(\frac{1}{2} C[\tau] R(g^{\mu\nu}) - \int d^2y (1-y) \left[y \nabla_\mu \tau \beta^I(\lambda^I(\mu e^{y\tau})) \partial_I c(\lambda^I(\mu e^{y\tau})) \tau \nabla^\mu \tau + c(\lambda^I(\mu e^{y\tau})) (\nabla \tau)^2 \right] \right) \\ &= \int d^2x \sqrt{-g} \left(+ \frac{1}{2} C[\tau] R(g^{\mu\nu}) - \int d^2y (1-y) \left[y \frac{d}{dy} c(\lambda^I(\mu e^{y\tau})) + c(\lambda^I(\mu e^{y\tau})) \right] (\nabla \tau)^2 \right) \end{aligned}$$

where we have defined:

$$C[\tau] = \int_\mu^{\mu e^\tau} c(\bar{\mu}) d(\ln \bar{\mu}) \quad (1.2.82)$$

Indeed, this follows immediately from the change of variables $\bar{\mu} = \lambda^I(\mu e^{y\tau})$.

The second takes the form:

$$\begin{aligned}
& - \int d^2x \int d^2y \sqrt{-\hat{g}[y](x)} \tau(x) \frac{1}{2} \chi_{IJ}(\lambda^I(\mu e^{y\tau})) \hat{g}^{\mu\nu}[y](x) \nabla_\mu \lambda^I(\mu e^{y\tau}) \nabla_\nu \lambda^J(\mu e^{y\tau}) \\
& = - \int d^2x \int d^2y \sqrt{-g} e^{-2(1-y)\tau} \tau(x) \frac{1}{2} \chi_{IJ}(\lambda^I(\mu e^{y\tau})) e^{2(1-y)\tau} g^{\mu\nu} \left[y \nabla_\mu \tau \beta^I(\lambda^I(\mu e^{y\tau})) \right] \left[y \nabla_\nu \tau \beta^J(\mu e^{y\tau}) \right] \\
& = - \int d^2x \int d^2y \sqrt{-g} y^2 \tau(x) \frac{1}{2} \chi_{IJ}(\lambda^I(\mu e^{y\tau})) \beta^I(\lambda^I(\mu e^{y\tau})) \beta^J(\mu e^{y\tau}) (\nabla \tau)^2 \\
& = - \int d^2x \int d^2y \sqrt{-g} y^2 \tau(x) \frac{1}{2} \beta^I(\lambda^I(\mu e^{y\tau})) \partial_I \tilde{c}(\lambda^I(\mu e^{y\tau})) (\nabla \tau)^2 \\
& = - \int d^2x \int d^2y \sqrt{-g} \frac{1}{2} y^2 \frac{d}{dy} \tilde{c}(\lambda^I(\mu e^{y\tau})) (\nabla \tau)^2
\end{aligned}$$

where we have used the gradient flow formula previously demonstrated:

$$\beta^I \partial_I \tilde{c} = \chi_{IK} \beta^I \beta^K \quad (1.2.83)$$

Finally, the last term turns out to be:

$$\begin{aligned}
& - \int d^2x \int d^2y \sqrt{-\hat{g}[y](x)} \hat{g}^{\mu\nu}[y](x) \nabla_\mu \tau(x) w_I(\lambda^I(\mu e^{y\tau})) \nabla_\nu \lambda^I \\
& = - \int d^2x \int d^2y \sqrt{-g} e^{-2(1-y)\tau} e^{2(1-y)\tau} g^{\mu\nu} \nabla_\mu \tau w_I(\lambda^I(\mu e^{y\tau})) y \nabla_\nu \tau \beta^I(\lambda^I(\mu e^{y\tau})) \\
& = - \int d^2x \int d^2y \sqrt{-g} y w_I(\lambda^I(\mu e^{y\tau})) \beta^I(\lambda^I(\mu e^{y\tau})) (\nabla \tau)^2
\end{aligned}$$

Putting all the terms proportional to $(\nabla \tau)^2$ together gives:

$$\begin{aligned}
& \int d^2x \int d^2y \sqrt{-g} \left[-(1-y)y \frac{d}{dy} c - (1-y)c - \frac{1}{2} y^2 \frac{d}{dy} \tilde{c} - y w_I \beta^I \right] (\nabla \tau)^2 \\
& = - \int d^2x \int d^2y \sqrt{-g} \left[(1-y)y \frac{d}{dy} c + c - 2yc + y\tilde{c} + \frac{1}{2} y^2 \frac{d}{dy} \tilde{c} \right] (\nabla \tau)^2 \\
& = - \int d^2x \int d^2y \sqrt{-g} \left\{ \frac{d}{dy} [(1-y)yc] + \frac{1}{2} \frac{d}{dy} [y^2 \tilde{c}] \right\} (\nabla \tau)^2 \\
& = - \int d^2x \sqrt{-g} \frac{1}{2} \tilde{c}(\lambda(\mu e^\tau)) (\nabla \tau)^2
\end{aligned}$$

We have therefore shown that:

$$\Gamma_{local}[g, \tau] = \int d^2x \sqrt{-g} \left(\frac{1}{2} C[\tau] R(g^{\mu\nu}) - \frac{1}{2} \tilde{c}(\lambda(\mu e^\tau)) (\nabla \tau)^2 \right) \quad (1.2.84)$$

* * * * *

In the above separated section, we prove that using (1.2.73) the local part of the dilaton effective action takes the form:

$$\Gamma_{local}[g, \tau] = \int d^2x \sqrt{-g} \left(\frac{1}{2} C[\tau] R(g^{\mu\nu}) - \frac{1}{2} \tilde{c}(\lambda(\mu e^\tau)) (\nabla \tau)^2 \right) \quad (1.2.85)$$

If we now take $g^{\mu\nu} = \eta^{\mu\nu}$, we end up with:

$$\Gamma_{\text{local}}[\eta, \tau] = - \int d^2x \frac{1}{2} \tilde{c}(\lambda(\mu e^\tau)) (\nabla \tau)^2 \quad (1.2.86)$$

This result was in some sense expected, or at least wished for. We will come to why that is in section 1.2.1.6.

Before doing so, let us also compute $\Gamma_{\text{non-local}}$. By definition, we have:

$$\Gamma_{\text{non-local}}[\eta, \tau] = \mathcal{W}[\mathcal{J}_2] = \mathcal{W}[\lambda[\tau, 1]] \quad (1.2.87)$$

where we have already taken the $g^{\mu\nu} = \eta^{\mu\nu}$. We then can write:

$$\mathcal{W}[\lambda[\tau, 1]] =: \exp \left\{ \int d^2x (\lambda[\tau, 1] - \lambda^I(\mu)) \frac{\delta}{\delta \bar{\lambda}^I(x)} \right\} : \mathcal{W}[\bar{\lambda}] \Big|_{\bar{\lambda}=\lambda(\mu)} \quad (1.2.88)$$

where the $: \dots :$ represent the fact that the derivatives do act on the λ s. This is simply the functional equivalent of a Taylor expansion, which can be written as:

$$f(x) =: \exp \left((x - y) \frac{d}{dy} \right) : f(y) \quad (1.2.89)$$

also in the better known case of standard functions. What we ultimately will be interested in is the expansion of $\Gamma_{\text{non-local}}$ in powers of the canonical dilaton ϕ which is defined as:

$$e^{-\tau} = 1 + \phi \quad (1.2.90)$$

* * * * *

We have:

$$\lambda^I(\mu e^\tau) = \lambda^I \left(\frac{\mu}{1 + \phi} \right) \quad (1.2.91)$$

We then obtain:

$$\frac{d}{d\phi} \lambda^I \left(\frac{\mu}{1 + \phi} \right) = - \frac{1}{1 + \phi} \beta^I \left(\lambda^I \left(\frac{\mu}{1 + \phi} \right) \right) \quad (1.2.92)$$

$$\frac{d^2}{d\phi^2} \lambda^I \left(\frac{\mu}{1 + \phi} \right) = \frac{1}{(1 + \phi)^2} \left[\beta^I \left(\lambda \left(\frac{\mu}{1 + \phi} \right) \right) + \beta^J \left(\lambda \left(\frac{\mu}{1 + \phi} \right) \right) \partial_J \beta^I \left(\lambda \left(\frac{\mu}{1 + \phi} \right) \right) \right] \quad (1.2.93)$$

We therefore have obtained that:

$$\begin{aligned} \Gamma_{\text{non-local}}[\eta, \tau] &= \mathcal{W}[\lambda(\mu)] + \int d^2x \left(-\phi \beta^I \frac{\delta}{\delta \bar{\lambda}^I} \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} + \frac{\phi^2}{2} \beta^J \left[\delta_J^I + \partial_J \beta^I \right] \frac{\delta}{\delta \bar{\lambda}^I(x)} \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \right) \\ &\quad + \int d^2x \int d^2y \frac{\phi(y)\phi(x)}{2} \beta^I \beta^J \frac{\delta}{\delta \bar{\lambda}^I(x)} \frac{\delta}{\delta \bar{\lambda}^J(y)} \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \end{aligned} \quad (1.2.94)$$

$$\begin{aligned} &= \mathcal{W}[\lambda(\mu)] - \int d^2x \phi(x) \beta^I \frac{\delta}{\delta \bar{\lambda}^I(x)} \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \\ &\quad + \int d^2x \int d^2y \frac{\phi(y)\phi(x)}{2} \left[\delta(x - y) \beta^J \left(\delta_J^I + \partial_J \beta^I \right) \frac{\delta}{\delta \bar{\lambda}^I(x)} + \beta^I \beta^J \frac{\delta}{\delta \bar{\lambda}^I(x)} \frac{\delta}{\delta \bar{\lambda}^J(y)} \right] \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \end{aligned} \quad (1.2.95)$$

* * * * *

A computation then yields that to second order in ϕ , we have:

$$\begin{aligned} \Gamma_{\text{non-local}}[\eta, \tau] = & \mathcal{W}[\lambda(\mu)] - \int d^2x \phi(x) \beta^I \frac{\delta}{\delta \lambda^I(x)} \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \\ & + \int d^2x \int d^2y \frac{\phi(y)\phi(x)}{2} \left[\delta(x-y) \beta^J \left(\delta_J^I + \partial_J \beta^I \right) \frac{\delta}{\delta \bar{\lambda}^I(x)} + \beta^I \beta^J \frac{\delta}{\delta \bar{\lambda}^I(x)} \frac{\delta}{\delta \bar{\lambda}^J(y)} \right] \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \end{aligned} \quad (1.2.96)$$

We now finally have all the ingredients we need to prove the results about the properties of the RG flow we are interest in.

1.2.1.6 Wess-Zumino dilaton action

Before continuing, let us now make a little excursion back into conformal theories. This section will explain why the result of equation (1.2.86) was in some degree expected. We have shown and used that the variation of the generating functional of a CFT in a curved background is equal to the conformal anomaly. The natural question to ask is then what is the form of the part of the generating functional whose variation gives the anomaly. This part is called the Wess-Zumino dilaton action. Let us recall that by introducing the dilaton in the way we did, namely $\hat{g}^{\mu\nu} = e^{2\tau} g^{\mu\nu}$, we have two transformations:

$$\begin{aligned} \text{Weyl transformations: } & \tau \rightarrow \tau + \sigma \quad \& \quad g^{\mu\nu} \rightarrow g^{\mu\nu} \\ \text{"Gauge" transformations: } & \tau \rightarrow \tau - \alpha \quad \& \quad g^{\mu\nu} \rightarrow e^{2\alpha} g^{\mu\nu} \end{aligned}$$

We then have:

$$\begin{aligned} \Delta_\sigma \mathcal{W}[\hat{g}] = \Delta_\sigma^g \mathcal{W}[\hat{g}] = & - \int d^Dx \sqrt{-\hat{g}(x)} \langle T(x) \rangle \sigma(x) \\ \equiv & \int d^Dx \mathcal{A}_\sigma[\hat{g}] \\ \equiv & \int d^Dx \sqrt{-\hat{g}(x)} \sigma(x) \mathcal{A}[\hat{g}] \end{aligned}$$

where we have used the fact that we are now working with a “pure” CFT. So the variation of $\mathcal{W}[\hat{g}]$ gives the conformal anomaly *in terms of* $\hat{g}$. Notice that it is crucial to make the distinction between g and $\hat{g}$. Let us now do the separation which we already did once in section 1.2.1.1:

$$S[\Phi, \hat{g}] = S^{(1)}[\Phi, g] + S^{(2)}[\hat{g}] \quad \Rightarrow \quad \mathcal{W}[\hat{g}] = \mathcal{W}^{(1)}[g] + \mathcal{W}^{(2)}[g, \tau] \quad (1.2.97)$$

First, notice that $S^{(1)}[\Phi, g]$ depends only on g and not $\hat{g}$ because by definition we take it to be Weyl invariant and in our implementation of the Weyl transformation only τ is affected. Furthermore, we have that

$$S^{(2)}[\hat{g}] = \mathcal{W}^{(2)}[g, \tau] \quad (1.2.98)$$

for the simple reason that since $S^{(2)}[\hat{g}]$ does not depend on the dynamical fields, it can be factorized out of the path-integral. This separation takes $\mathcal{W}^{(1)}[g]$ to be Weyl invariant.

Of course, this separation is not unique since one can always move Weyl invariant terms between the two parts. But what we have obtained from this separation, is that:

$$\Delta_\sigma \mathcal{W}[\hat{g}] = \Delta_\sigma \mathcal{W}^{(2)}[g, \tau]$$

i.e. $\mathcal{W}^{(2)}[g, \tau]$ gives the conformal anomaly in terms of $\hat{g}$.

What happens under the gauge transformations? We obtain the following:

$$\begin{aligned} 0 &= \Delta_\alpha^G \mathcal{W}[\hat{g}] = \Delta_\alpha^G \mathcal{W}^{(1)}[g] + \Delta_\alpha^G \mathcal{W}^{(2)}[g, \tau] \\ \Rightarrow \Delta_\alpha^G \mathcal{W}^{(1)}[g] &= -\Delta_\alpha^G \mathcal{W}^{(2)}[g, \tau] \end{aligned}$$

Now notice two things. First that by definition we have taken $\mathcal{W}^{(1)}$ to be the generating functional of a CFT *in a curved background determined by g* , which therefore certainly has to reproduce the trace anomaly in terms of g under a Weyl transformation of g . Secondly, the “gauge” transformation described above implements from the point of view of g a Weyl transformation with parameter α . We therefore have the following results:

$$\Delta_\sigma \mathcal{W}[\hat{g}] = \Delta_\sigma \mathcal{W}^{(2)}[g, \tau] = \int d^D x \mathcal{A}_\sigma[\hat{g}] \quad (1.2.99)$$

$$\Delta_\alpha^G \mathcal{W}^{(2)}[g, \tau] = - \int d^D x \mathcal{A}_\sigma[g] \quad (1.2.100)$$

Notice here the crucial difference: The Weyl variation gives the trace anomaly in terms of $\hat{g}$ and the gauge variation gives minus the anomaly in terms of g .

The functional $\mathcal{W}^{(2)}$ is called the Wess-Zumino dilaton action. We now give its explicit form in two dimensions and four dimensions. For the more general result in every even even dimension, see [13].

1.2.1.6.1 In two dimensions We have:

$$\mathcal{W}_{2D}^{(2)}[g, \tau] = c \int d^2 x \sqrt{-g(x)} \left(\tau R - (\partial\tau)^2 \right) \quad (1.2.101)$$

Notice that curvature terms are a function of g not $\hat{g}$. Indeed, we have:

$$\begin{aligned} \Delta_\sigma \mathcal{W}_{2D}^{(2)}[g, \tau] &= c \int d^2 x \sqrt{-g(x)} (+\sigma R - (2\partial^\mu \tau \partial_\mu \sigma)) \\ &= c \int d^2 x \sqrt{-\hat{g}(x)} e^{2\tau} (\sigma R + 2\Box\tau\sigma) \\ &= c \int d^2 x \sqrt{-\hat{g}(x)} \sigma e^{2\tau} (R + 2\Box\tau) \\ &= c \int d^2 x \sqrt{-\hat{g}(x)} \sigma(x) \hat{R} \end{aligned}$$

and, to check our previous assertion, we have:

$$\begin{aligned} \Delta_\alpha^G \mathcal{W}_{2D}^{(2)}[g, \tau] &= \int d^2 x \sqrt{-g} \left[-2\alpha(\tau R - (\partial\tau)^2) + (-\alpha R + \tau(2\Box\alpha + 2R\alpha) - (2\alpha(\partial\tau)^2 - 2\partial_\mu \alpha \partial^\mu \tau)) \right] \\ &= -c \int d^2 x \sqrt{-g(x)} \alpha R \end{aligned}$$

One possible way to find the form the dilaton WZ action is to start with

$$\mathcal{W}_{2D}^{(2)}[g, \tau] = c \int d^2 x \sqrt{-g(x)} \tau R \quad (1.2.102)$$

and take its Weyl variation and then introduce new terms to compensate for the unwanted terms and then repeat the procedure until it stops.

It is now apparent from the form of $\mathcal{W}_{2D}^{(2)}$ why the result off criticality was expected. Indeed, for $g = \eta$, we have

$$\mathcal{W}_{2D}^{(2)}[\eta, \tau] = -c \int d^2x \sqrt{-g(x)} (\partial\tau)^2 \quad (1.2.103)$$

which is exactly the result we obtained but with $\tilde{c}$ depending on the renormalization scale and reducing to c at a conformal fixed point.

To relate our result off-criticality even better to the Wess-Zumino dilaton action, we can do the following:

$$C[\tau] = \int_{\mu}^{\mu e^{\tau}} c(\bar{\mu}) \frac{d\bar{\mu}}{\bar{\mu}} \approx \mu(1 + \tau)c(\mu) - \mu c(\mu) = \tau c(\mu) \quad (1.2.104)$$

So that we finally have:

$$\begin{aligned} \Gamma_{\text{local}}[g, \tau] &\approx \int d^2x \sqrt{-g} \left(\frac{1}{2} c(\lambda(\mu)) \tau R(g^{\mu\nu}) - \frac{1}{2} c(\lambda(\mu)) (\nabla\tau)^2 \right) \\ &= \frac{1}{2} \mathcal{W}_{2D}^{(2)}[g, \tau] \end{aligned} \quad (1.2.105)$$

1.2.1.6.2 In four dimensions For completeness and later convenience, we also give the Wess-Zumino term in four dimensions:

$$\Gamma_4[g, \tau] = \int d^4x \sqrt{-g(x)} \left[-a \left(\tau E_4 + 4G_{\mu\nu} \partial^\mu \tau \partial^\nu \tau - 4\Box\tau (\partial\tau)^2 + 2(\partial\tau)^2 \right) + c\tau W^2 \right] \quad (1.2.106)$$

where a and c are the anomaly coefficients in four dimensions the Euler density E_4 and the Weyl tensor squared W^2 respectively.

1.2.2 Dilaton “scattering” amplitude and properties of RG flows in 2D

1.2.2.1 The dilaton “scattering” amplitude

In this section, we will prove the non-perturbative weak a-theorem using what is called in a slight abuse of language the dilaton “scattering” amplitude. The word “scattering” is between quotation marks because the dilaton is a background field and therefore does not have any dynamics. The idea of using this dilaton scattering amplitude to study the problems at hand is due to Komargodski and Schwimmer in their papers [4] and [5]. In the case of 2D, the dilaton scattering amplitude (DSA) is a bit of an overkill, because it is trivially related to a single correlator of T , but in 4 dimensions, it was the crucial step towards a proof as we will see in the section about the a-theorem in 4 dimensions.

The idea is to study the two-point function of the canonical dilaton. We therefore define the DSA (in 2 dimensions) by:

$$(2\pi)^2 \delta(p_1 + p_2) A(p_1, p_2) = \mathcal{F} \left\{ \frac{\delta}{\delta\phi(x_1)} \frac{\delta}{\delta\phi(x_2)} \mathcal{W}[g, \phi] \Big|_{\substack{g=\eta \\ \phi=0}} \right\} (p_1, p_2) \quad (1.2.107)$$

where ϕ is defined in equation (1.2.90).

Let us now relate A to the correlators of T . To do so, notice that:

$$\frac{\delta}{\delta\phi(x)} = \int d^4y \frac{\delta\tau(y)}{\delta\phi(x)} \frac{\delta}{\delta\tau(y)} = -(1 + \phi)^{-1} \frac{\delta}{\delta\tau(x)} = -e^{\tau} \frac{\delta}{\delta\tau(x)} \quad (1.2.108)$$

We then have:

$$\begin{aligned}
\frac{\delta}{\delta\phi(x_1)} \frac{\delta}{\delta\phi(x_2)} \mathcal{W}[g, \phi] &= e^{\tau_1} \frac{\delta}{\delta\tau(x_1)} e^{\tau_2} \frac{\delta}{\delta\tau(x_2)} \mathcal{W}[g, \phi] \\
&= e^{\tau_1} e^{\tau_2} \left(\delta(x_1 - x_2) \frac{\delta}{\delta\tau(x_2)} \mathcal{W}[g, \phi] + \frac{\delta}{\delta\tau(x_1)} \frac{\delta}{\delta\tau(x_2)} \mathcal{W}[g, \phi] \right) \\
\Rightarrow \frac{\delta}{\delta\phi(x_1)} \frac{\delta}{\delta\phi(x_2)} \mathcal{W}[g, \phi] \Big|_{\substack{g=\eta \\ \phi=0}} &= \delta(x_1 - x_2) \langle T(x_2) \rangle + i \langle T(x_1) T(x_2) \rangle
\end{aligned} \tag{1.2.109}$$

Now since the one-point of T is zero (because we can absorb the contributions of the one-point functions into the cosmological constant term), we finally obtain:

$$(2\pi)^2 \delta(p_1 + p_2) A(p_1, p_2) = i \langle T(p_1) T(p_2) \rangle \tag{1.2.110}$$

This relation is essentially why the a-theorem in 2 dimensions has been proven 30 years before its equivalent in 4 dimensions. Indeed, the DSA is the crucial quantity in all that follows and in two dimensions it simply is the correlator of two T s, whereas in 4 dimensions, it is a highly non-trivial combination of 2-, 3- and 4-point functions of T .

Notice that generically, the two-point function of T is not finite and requires renormalization. Does there exist a condition under which the two-point function of T would automatically be finite? We proved in section 1.2.1.4 that the RG behavior is completely controlled by the anomaly. Indeed, we recall that we have proven that:

$$\Delta^{RG} \mathcal{W} = \int d^2x \mathcal{A}_{-1} \tag{1.2.111}$$

Therefore the question becomes whether there exists a condition such that the anomaly (or the integral of the anomaly) vanishes in pure dilaton background (i.e. $g = \eta$ and $\lambda^I = \text{const}$). From the equation above, this would then directly imply the RG independence of $\mathcal{W}$ and by extension that of the amplitude. In two dimensions, the anomaly is then proportional to the Ricci scalar $\hat{R}$, who plays the role of the Euler density in two dimensions, and therefore integrates to zero in topologically trivial spaces. We therefore do not have to impose any conditions on ϕ to guarantee finiteness of the DSA.

As a consequence of the above discussion, dimensional analysis implies that the DSA takes the form:

$$A(p_1) = p_1^2 F\left(\frac{p_1^2}{\mu^2}, \lambda(\mu)\right) + \Lambda_2 \tag{1.2.112}$$

where F is a function which is invariant under the renormalization group and Λ_2 is the cosmological constant. We now have all the ingredients to proceed with the various proofs.

1.2.2.2 The non-perturbative weak a-theorem

We start by giving a non-perturbative proof for the weak a-theorem. The assumption is that we have a UV CFT which is deformed by relevant operators, which then generate an RG flow towards another IR CFT fixed point. Let us start by considering the UV fixed point. We then very generally write:

$$S = S_{UV} + \int d^2x \sqrt{-\hat{g}} \sum_i c_i m^{2-\Delta_i} \hat{\mathcal{O}}_i \tag{1.2.113}$$

where S_{UV} is the action of the UV CFT, $\hat{\mathcal{O}}_i$ relevant and marginal primary operators which therefore have dimension $\Delta_i \leq 2$, m is the mass scale of the associated flow, which

for simplicity we assume to be the unique characteristic mass scale controlling the flow and the c_i are dimensionless coefficients. We have to separate two cases: the marginal operators and the strictly relevant operators. We already know what the effect of the marginal operators will be. Indeed, one of the reasons to study the local RG equation is because it answers that specific question. More precisely, we have proven that in the presence of marginal deformations the generating functional separates into a local and a non-local contribution given in equations (1.2.86) and (1.2.96) and the DSA then takes the form (1.2.112). In the case of the strictly relevant operators, since we are only interested in the leading behavior very close to the fixed point, dimensional analysis is an adequate guide.

Let us now consider the dilaton couplings arising from the relevant deformations. We had written the original action in terms of the fields of the CFT denoted by $\hat{\Phi}$ and the background metric. If we now define (in addition to the definition of $\hat{g}$ given previously):

$$\hat{\Phi} = e^{\tau\Delta_\Phi} \Phi \quad (1.2.114)$$

then the relevant deformations take the form:

$$S_{rel}[\hat{\Phi}, \hat{g}] = \int d^2x \sqrt{-g} (me^{-\tau})^{2-\Delta} \mathcal{O}[\Phi, g] \quad (1.2.115)$$

where the power of two came from the metric determinant and the power of Δ from the combination of $\hat{\Phi}$ in $\mathcal{O}_i$.

* * * * *

Let us now compute these contributions to the DSA:

$$\begin{aligned} \frac{\delta}{\delta\phi(x_1)} \frac{\delta}{\delta\phi(x_2)} \Gamma_{local}[\eta, \phi] \Big|_{\phi=0} &= \frac{\delta}{\delta\phi(x_1)} \frac{\delta}{\delta\phi(x_2)} \left(\int d^2x \left(-\frac{1}{2}\right) \tilde{c} \left(\lambda \left(\frac{\mu}{1+\phi} \right) \right) (\partial\phi)^2 \frac{1}{1+\phi^2} \right) \Big|_{\phi=0} \\ &= -\tilde{c}(\lambda(\mu)) \int d^2x \partial_\mu^x \delta(x-x_1) \partial_x^\mu \delta(x-x_2) \end{aligned} \quad (1.2.116)$$

$$\Rightarrow A_{local}(p_1) = (p_1 \cdot p_2) \tilde{c}(\lambda(\mu)) = -p_1^2 \tilde{c}(\lambda(\mu)) \quad (1.2.117)$$

This is trivial to see, since at two functional derivative level, they must act on the derivatives of ϕ , all the other terms cancel after taking $\phi = 0$. We also have:

$$\begin{aligned} \frac{\delta}{\delta\phi(x_1)} \frac{\delta}{\delta\phi(x_2)} \Gamma_{non-local}[\eta, \phi] \Big|_{\phi=0} &= \int d^2x \int d^2y \delta(x_1-x) \delta(x_2-y) \\ &\quad \times \left[\delta(x-y) \beta^J \left(\delta_J^I + \partial_J \beta^I \right) \frac{\delta}{\delta \bar{\lambda}^I(x)} + \beta^I \beta^J \frac{\delta}{\delta \bar{\lambda}^I(x)} \frac{\delta}{\delta \bar{\lambda}^J(y)} \right] \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \end{aligned} \quad (1.2.118)$$

$$\Rightarrow (2\pi)^2 \delta(p_1 + p_2) A_{non-local}(p_1) = \beta^J \left(\delta_J^I + \partial_J \beta^I \right) \langle \mathcal{O}_J(p_1 + p_2) \rangle + \beta^I \beta^J i \langle \mathcal{O}_I(p_1) \mathcal{O}_J(p_2) \rangle$$

$$\Rightarrow (2\pi)^2 \delta(p_1 + p_2) A_{non-local}(p_1) = \beta^J \left(\delta_J^I + \partial_J \beta^I \right) \langle \mathcal{O}_J(0) \rangle + \beta^I \beta^J i \langle \mathcal{O}_I(p_1) \mathcal{O}_J(-p_1) \rangle \quad (1.2.119)$$

The one-point function term will give zero (we have already argued this in a previous section), whereas the second term is of order β^2 and becomes subleading very close to the conformal fixed point. A better argument is that one can add a local functional of the background sources to $\mathcal{W}$ such that the order β^2 is canceled. We will see this in much more detail in the next section.

For the relevant terms, simple dimensional analysis tells us that:

$$A_{rel}(p_1) \propto (m^2)^{2-\Delta} (p_1^2)^{\Delta-1} \propto p_1^2 \left(\frac{m^2}{p_1^2} \right)^{2-\Delta} \quad (1.2.120)$$

which is therefore clearly subleading since $\Delta < 2$.

* * * * *

It is proven in the above separated section that for the UV theory, the leading contribution to the DSA reduces to:

$$A(p_1) \xrightarrow{\frac{p_1^2}{m^2} \rightarrow \infty} -p_1^2 c_{UV} \left(1 + \left(\frac{m^2}{p_1^2} \right)^{2-\Delta_{UV}} \right) + \Lambda_{cc}^{UV} \quad (1.2.121)$$

where Δ_{UV} is the largest of the operator dimensions and we used the fact that $\tilde{c}$ reduces to c at the fixed point.

In the IR, a similar argument yields:

$$A(p_1) \xrightarrow{\frac{p_1^2}{m^2} \rightarrow 0} -p_1^2 c_{IR} \left(1 + \left(\frac{p_1^2}{m^2} \right)^{\Delta_{IR}-2} \right) + \Lambda_{cc}^{IR} \quad (1.2.122)$$

To conclude, we will now use a dispersion relation. Let us define:

$$\eta_m \equiv \frac{p_1^2}{m^2} \quad (1.2.123)$$

The Cauchy theorem then implies:

$$\frac{1}{2\pi i} \oint_C \frac{A(\eta_m)}{\eta_m^2} d\eta_m = 0 \quad (1.2.124)$$

where the contour is shown in figure 1.1. The inner circle of the contour is taken with a radius $\varepsilon \rightarrow 0$ and the two paths parallel to the branch cut are taken at a displacement of $\pm i\delta$ respectively of the real positive axis.

We then obtain:

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_0^{2\pi} i d\theta r e^{i\theta} \left(\frac{-m^2 c_{UV}}{r e^{i\theta}} + \frac{\Lambda_{cc}^{UV}}{r^2 e^{2i\theta}} \right) \quad \text{where} \quad \eta = r e^{i\theta} \\ &= -m^2 c_{UV} + \frac{i \Lambda_{cc}^{UV}}{2\pi i} \int_0^{2\pi} e^{-i\theta} d\theta \\ &= -m^2 c_{UV} \end{aligned} \quad (1.2.125)$$

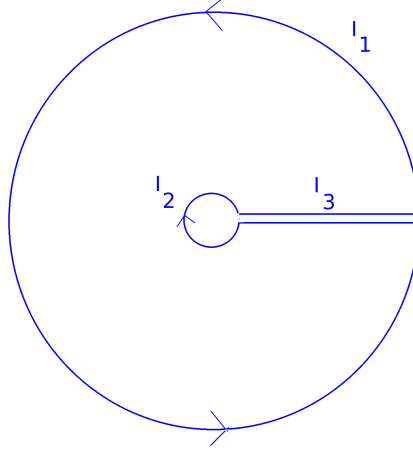


Figure 1.1: Contour for the dispersion relation.

In exactly the same way, we obtain:

$$I_2 = +m^2 c_{IR} \quad (1.2.126)$$

The last part of the contour gives us:

$$\begin{aligned} I_3 &= \frac{1}{2\pi i} \int_{\infty}^0 \frac{A(\eta_m - i\delta)}{\eta_m^2} d\eta_m + \frac{1}{2\pi i} \int_0^{\infty} \frac{A(\eta_m + i\delta)}{\eta_m^2} d\eta_m \\ &= \frac{1}{2\pi i} \int_0^{\infty} \frac{A(\eta_m + i\delta) - A(\eta_m - i\delta)}{\eta_m^2} d\eta_m \\ &= \frac{1}{2\pi i} \int_0^{\infty} \frac{A(\eta_m + i\delta) - A(\eta_m + i\delta)^*}{\eta_m^2} d\eta_m \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} A(\eta_m)}{\eta_m^2} d\eta_m \end{aligned} \quad (1.2.127)$$

where we used the Schwarz reflection principle, which states that if we have an analytic function which is real on the real axis, we can analytically continue it as $A(p_1^{2*}) = A(p_1^2)^*$. So that the final result is:

$$c_{UV} - c_{IR} = \frac{1}{m^2 \pi} \int_0^{\infty} \frac{\text{Im} A(\eta_m)}{\eta_m^2} d\eta_m \quad (1.2.128)$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} (i \langle T(p_1) T(-p_1) \rangle)}{(p_1)^2} dp_1^2 \quad (1.2.129)$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\text{Re} \langle T(p_1) T(-p_1) \rangle}{(p_1)^2} dp_1^2 \quad (1.2.130)$$

One possible way of proving that $\text{Im} (i \langle T(p_1) T(-p_1) \rangle) \geq 0$ is given in the appendix B.2, where it is also mentioned that $\text{Im} (i \langle T(p_1) T(-p_1) \rangle)$ is essentially the spectral density for the two-point function of T .

Using this positivity, we have therefore finally proven the weak non-perturbative a-theorem.

1.2.2.3 The perturbative strong a-theorem or Zamolodchikov's theorem

1.2.2.3.1 The original argument For comparison, we present in this section Zamolodchikov's original argument as presented by Cardy[14]. We consider 2D field theory. One

can show that by rotational and translational invariance of the theory, we have:

$$\langle T(z, \bar{z})T(0, 0) \rangle = \frac{F(\mu^2|z|^2)}{z^4} \quad (1.2.131)$$

$$\begin{aligned} \langle T(z, \bar{z})\Theta(0, 0) \rangle &= \langle \Theta(z, \bar{z})T(0, 0) \rangle \\ &= \frac{G(\mu^2|z|^2)}{z^3\bar{z}} \end{aligned} \quad (1.2.132)$$

$$\langle \Theta(z, \bar{z})\Theta(0, 0) \rangle = \frac{H(\mu^2|z|^2)}{z^2\bar{z}^2} \quad (1.2.133)$$

where we have defined $T \equiv T_{zz}$ and $\Theta \equiv 4T_{z\bar{z}} = T_\mu^\mu$. When we write that the various functions depend on $\mu^2|z|^2$ it would be more accurate to say that they depend on $\ln(\mu^2|z|^2)$. What we are going to be interested in is the change of the above quantities when we change the length scales:

$$\dot{f}(\mu^2|z|^2) \equiv \frac{d}{d \ln |z|^2} f(\mu^2|z|^2) = z \partial_z f(\mu^2|z|^2) = \bar{z} \partial_{\bar{z}} f(\mu^2|z|^2) \quad (1.2.134)$$

We now use the conservation of the stress tensor which states (see appendix A.1):

$$\partial_{\bar{z}} T + \frac{1}{4} \partial_z \Theta = 0 \quad (1.2.135)$$

which then imply:

$$\begin{aligned} \partial_{\bar{z}} \langle T(z, \bar{z})T(0, 0) \rangle &= \frac{1}{z^4} \partial_{\bar{z}} F = \frac{1}{z^4 \bar{z}} \dot{F} \\ \partial_z \langle \Theta(z, \bar{z})T(0, 0) \rangle &= \frac{1}{z^4 \bar{z}} (\dot{G} - 3G) \\ \partial_{\bar{z}} \langle \Theta(z, \bar{z})T(0, 0) \rangle &= \frac{1}{\bar{z}^2 z^3} (\dot{G} - G) \\ \partial_z \langle \Theta(z, \bar{z})\Theta(0, 0) \rangle &= \frac{1}{\bar{z}^2 z^3} (\dot{H} - 2H) \end{aligned}$$

Combining these equations then yields, the following two equations:

$$\begin{aligned} \dot{F} + \frac{1}{4} (\dot{G} - 3G) &= 0 \\ \dot{G} - G + \frac{1}{4} (\dot{H} - 2H) &= 0 \\ \Rightarrow \quad \dot{C} &= -\frac{3}{4} H \end{aligned} \quad (1.2.136)$$

where we have defined:

$$C = 2F - G - \frac{3}{8} H \quad (1.2.137)$$

We can now conclude using the positivity of $\langle \Theta(z, \bar{z})\Theta(0, 0) \rangle$ (see appendix B.2) that H is positive and therefore the function C is monotonously decreasing along the RG flow. Specifically, we have that as the distance increases, or equivalently the energy decreases, C decreases. We therefore have proven the strong a-theorem and since C reduces to the central charge at fixed points, we also have a perturbative proof of the c-theorem.

1.2.2.3.2 Argument using the local RG equation and the dilaton scattering amplitude We already have nearly everything. The only thing which is missing is the positive definiteness of χ_{IJ} for which we have already provided one simple argument. We now provide another, maybe more rigorous, one.

We have already seen in the previous section that we can then parametrize the DSA as:

$$A(p^2) = p^2 F\left(\frac{p^2}{\mu^2}, \lambda(\mu)\right) + \Lambda \equiv -p^2 \alpha(p^2) + \Lambda \quad (1.2.138)$$

We can now write a dispersion relation:

$$\frac{1}{2\pi i} \oint_C \frac{A(p^2)}{(p^2)^2} dp^2 = 0 \quad (1.2.139)$$

where the contour is now the same as in figure 1.1 except for the fact the radii are now taken to be p^1 and p^2 instead of 0 and ∞ . Let us also define the “average amplitude”:

$$\bar{\alpha}(p^2) = \frac{1}{2\pi} \int_0^{2\pi} \alpha(p^2 e^{i\theta}) d\theta \quad (1.2.140)$$

Exactly the same computation as before then yields:

$$\bar{\alpha}((p_2)^2) - \bar{\alpha}((p_1)^2) = \frac{1}{\pi} \int_{(p_1)^2}^{(p_2)^2} \frac{dp^2}{(p^2)^2} \left(-\text{Im} \alpha(p^2) \right) \geq 0 \quad (1.2.141)$$

Two things need to be emphasized at this point: By the Schwarz reflection principle, $\bar{\alpha}$ is real and since the amplitude is finite, the difference also has to be finite. Let us study both sides of equation (1.2.141).

Concerning the left hand side, we have already proven in the previous section that:

$$\alpha_{\text{local}}(p^2) = \tilde{c}(\lambda(\mu)) \quad \& \quad \alpha_{\text{non-local}}(p^2) = O(B^2) \quad (1.2.142)$$

It is at this point that the scheme dependence we already alluded to in a previous section really comes in handy. Indeed, this scheme dependence allows us to add a local functional of the sources to the generating functional. If we add the term

$$-\frac{1}{2} b R \quad (1.2.143)$$

to the generating functional, it clearly does not affect the DSA since the latter is computed in flat space but it does generate a transformation of $\tilde{c}$:

$$\tilde{c} \rightarrow \tilde{c} + \beta^I \partial_I b \quad (1.2.144)$$

while χ_{IJ} stays untouched. By choosing an appropriate b (which is a function of the couplings and by extension possibly of the β functions), we can find a scheme in which

$$\bar{\alpha} = \tilde{c} \quad (1.2.145)$$

Notice that both parts of α are independent of p^2 , so that we finally obtain that:

$$\bar{\alpha}(p^2) = \tilde{c}(\lambda(\mu)) = \tilde{c}(\lambda(\sqrt{p^2})) \quad (1.2.146)$$

where in the second equality we used the fact that the amplitude is μ independent.

On the other hand, the imaginary of α can only come from the non-local part of the dilaton effective action.

We have:

$$\begin{aligned} (2\pi)^2 \delta(p_1 + p_2) \left[-\text{Im}\alpha(p^2) \right] &= \text{Im} \left(i\beta^I \beta^J \langle \mathcal{O}_I(p) \mathcal{O}_J(-p) \rangle \right) \\ &= \beta^I \beta^J \text{Re} \langle \mathcal{O}_I(p) \mathcal{O}_J(-p) \rangle \end{aligned}$$

Let us adapt the method used in appendix B.2. We have:

$$\text{Re} \langle \mathcal{O}_I(x_1) \mathcal{O}_J(x_2) \rangle = \langle \mathcal{O}_I(x_1) \mathcal{O}_J(x_2) \rangle + \langle \mathcal{O}_J(x_2) \mathcal{O}_I(x_1) \rangle \quad (1.2.147)$$

Then using

$$\langle \Omega | \mathcal{O}(x) | \psi \rangle = e^{-ipx} \langle \Omega | \mathcal{O}(0) | \psi \rangle \quad (1.2.148)$$

we obtain:

$$\begin{aligned} \text{Re} \langle \mathcal{O}_I(p_1) \mathcal{O}_J(p_2) \rangle &= \int dxe^{i(p_1 x_1 + p_2 x_2)} \left(\langle \mathcal{O}_I(x_1) \mathcal{O}_J(x_2) \rangle + \langle \mathcal{O}_J(x_2) \mathcal{O}_I(x_1) \rangle \right) \\ &= (2\pi)^2 \delta(p_1 + p_2) \left((2\pi)^2 \delta(p_1 - p_\psi) \langle \Omega | \mathcal{O}_I(0) | \psi \rangle \langle \Omega | \mathcal{O}_J(0) | \psi \rangle^* \right. \\ &\quad \left. + (2\pi)^2 \delta(p_1 + p_\psi) \langle \Omega | \mathcal{O}_J(0) | \psi \rangle \langle \Omega | \mathcal{O}_I(0) | \psi \rangle^* \right) \\ &= (2\pi)^2 \delta(p_1 + p_2) (2\pi)^2 \delta(p_1 - p_\psi) \langle \Omega | \mathcal{O}_I(0) | \psi \rangle \langle \Omega | \mathcal{O}_J(0) | \psi \rangle^* \quad (1.2.149) \end{aligned}$$

where in the last equality we have used that the p_ψ are physical.

The above computation proves that one can write:

$$-\text{Im}\alpha(p^2) = \frac{1}{p^2} \sum_{\psi} (2\pi)^2 \delta(p - p_\psi) \beta^I \beta^J \mathcal{M}_I(\psi) \mathcal{M}_J(\psi)^* \quad (1.2.150)$$

where we have defined:

$$\beta^I \mathcal{M}_I(\psi) \equiv \beta^I \langle \Omega | \mathcal{O}_I(0) | \psi \rangle \quad (1.2.151)$$

We thus have proven the following formula:

$$-\text{Im}\alpha(p^2) = \beta^I \beta^J G_{IJ} \quad (1.2.152)$$

where

$$G_{IJ} = \frac{1}{p^2} \sum_{\psi} (2\pi)^2 \delta(p_\psi - p) \langle \Omega | \mathcal{O}_I(0) | \psi \rangle \langle \psi | \mathcal{O}_J(0) | \Omega \rangle \quad (1.2.153)$$

We can now plug this result back into our original dispersion relation which yields:

$$\bar{\alpha}((p_2)^2) - \bar{\alpha}((p_1)^2) = \frac{1}{\pi} \int_{(p_1)^2}^{(p_2)^2} \frac{dp^2}{p^2} \beta^I \beta^J G_{IJ} \quad (1.2.154)$$

To connect G_{IJ} and χ_{IJ} , let us start by integrating the gradient flow equation (1.2.43) which yields:

$$\tilde{c}(\lambda(\mu_2)) - \tilde{c}(\lambda(\mu_1)) = \int_{\mu_1}^{\mu_2} \chi_{IJ}(\lambda(\mu)) \beta^I(\lambda(\mu)) \beta^J(\lambda(\mu)) \frac{d\mu}{\mu} \quad (1.2.155)$$

Now doing the change of variables $\mu = \sqrt{p^2}$ (which essentially amounts to choosing the renormalization scale) then gives:

$$\tilde{c}(\lambda(\sqrt{(p_2)^2})) - \tilde{c}(\lambda(\sqrt{(p_1)^2})) = \int_{(p_1)^2}^{(p_2)^2} \chi_{IJ}(\lambda(\sqrt{p^2})) \beta^I(\lambda(\sqrt{p^2})) \beta^J(\lambda(\sqrt{p^2})) \frac{dp^2}{2p^2} \quad (1.2.156)$$

Identifying the right hand side of both equations then implies:

$$\chi_{IJ} = \frac{2}{\pi} G_{IJ} + \Delta_{IJ} \quad (1.2.157)$$

where Δ_{IJ} is a matrix such that

$$\Delta_{IJ} \beta^I \beta^J = 0 \quad (1.2.158)$$

We have therefore proven that χ_{IJ} is positive definite in perturbation theory up to the Δ matrix.

The first consequence is the perturbative strong a-theorem. Indeed, the gradient flow formula (1.2.43) takes the form:

$$\mu \frac{d}{d\mu} \tilde{c} \equiv \beta^I \partial_I \tilde{c} = \chi_{IJ} \beta^I \beta^J \quad (1.2.159)$$

which, due to the positive definiteness of χ_{IJ} , directly implies that $\tilde{c}$ is the monotonously increasing function we were looking for.

We have already mentioned that

$$\tilde{c}(\lambda(\mu_2)) - \tilde{c}(\lambda(\mu_1)) = \int_{\mu_1}^{\mu_2} \chi_{IJ}(\lambda(\mu)) \beta^I(\lambda(\mu)) \beta^J(\lambda(\mu)) \frac{d\mu}{\mu} \quad (1.2.160)$$

has to be finite as long as the flow stays in the perturbative neighborhood of the fixed point. This will yield a straightforward bound on the RG flow asymptotics. Specifically, if the flow stays in the mentioned neighborhood asymptotically as $\ln \mu \rightarrow \pm\infty$, the *positive* integrand has to vanish, i.e.

$$\lim_{\ln \mu \rightarrow \pm\infty} \chi_{IJ}(\lambda(\mu)) \beta^I(\lambda(\mu)) \beta^J(\lambda(\mu)) = 0 \quad (1.2.161)$$

There are two conditions under which this can happen:

- The matrix $\chi_{IJ}(\lambda(\mu))$ tends to a matrix with vanishing eigenvalues. This possibility can be excluded because it is well-known that at a fixed point χ_{IJ} is essentially unity and if the deformations transform the eigenvalues to zero, then our perturbativity assumption does not hold anymore.
- The beta function β^I vanish asymptotically.

We have therefore proven that under the assumption that the flow stays perturbative, its asymptotics are characterized by a vanishing beta function, i.e. the asymptotics of the flow have to be CFTs. More precisely, since we are considering a simplified case, one could be tempted to say that we have only proven that the asymptotics are an SFT. It turns out that the more general treatment implies that the asymptotics indeed have to be CFTs.

1.3 The a-theorem(s) in 4D

1.3.1 The local RG equation

In the previous section about the local RG equation in two dimensions, we introduced most of the tools and techniques. The difference in this section about the local RG equation in four dimension is essentially that we will consider more terms which simply make the calculations more messy than they already are. It is worth emphasizing that conceptually most ingredients are already present in the previous section. This is

maybe overly simplistic since we are going to consider for instance insertions of currents which introduce some subtleties which turn out to be crucial. This section is essentially a summary of [7] and the reader is strongly encouraged to consult that paper for more details. There is no originality in this section, it is a report of our understanding of the local RG equation in four dimensions.

1.3.1.1 General setup

The situation we consider is as before. We deform a CFT by turning on marginal deformations with coupling constants λ^I associated to scalar operators $\mathcal{O}_I$ which at the fixed point are primaries with dimension 4. The difference with the discussed two dimensional case, is that we also assume that the fixed point has an exact flavor symmetry G_F , which will be broken explicitly once the couplings are turned on. The goal of the procedure is to rewrite the correlators of T in term of complete basis of scalar operators of dimension 4. Dimensional analysis then tells us that in addition to the standard marginal operators $\mathcal{O}_I$, there could also be divergences of currents $\nabla_\mu J^\mu$ and operators of the form $\square \mathcal{O}_a$, where at the fixed point, $\mathcal{O}_a$ correspond to primaries of dimension two.

The idea is to then introduce sources for these operators in the form of background fields. Indeed, the EM tensor will be sourced by a background metric $g^{\mu\nu}(x)$, $\mathcal{O}_I$ by $\lambda^I(x)$, J_A^μ by the background vector fields $A_\mu^A(x)$ which gauge the flavor group G_F . For simplicity, we are going to ignore the dimension 2 operators and only mention in passing what effect these would have. As before, we indicate the collection of sources by $\mathcal{J} \equiv (g^{\mu\nu}, \lambda^i, A_\mu^A)$. We then introduce the *renormalized* partition function, which is a functional of the sources:

$$\mathcal{Z}[\mathcal{J}] = e^{i\mathcal{W}[\mathcal{J}]} = \int \mathcal{D}[\Phi] e^{iS[\Phi, \mathcal{J}]} \quad (1.3.1)$$

The definition of the the correlators is as before:

$$\begin{aligned} \langle T(x_1) \cdots T(x_n) \rangle &= \frac{(-i)^{n-1} 2^n}{\sqrt{-g(x_1)} \cdots \sqrt{-g(x_n)}} g^{\mu_1 \nu_1}(x_1) \frac{\delta}{\delta g^{\mu_1 \nu_1}(x_1)} \\ &\quad \times g^{\mu_2 \nu_2}(x_2) \frac{\delta}{\delta g^{\mu_2 \nu_2}(x_2)} \cdots g^{\mu_n \nu_n}(x_n) \frac{\delta}{\delta g^{\mu_n \nu_n}(x_n)} \mathcal{W}[\mathcal{J}] \Big|_{\mathcal{J}=0} \end{aligned} \quad (1.3.2)$$

$$\langle \mathcal{O}_{I_1}(x_1) \cdots \mathcal{O}_{I_n}(x_n) \rangle = \frac{(-i)^{n-1}}{\sqrt{-g(x_1)} \cdots \sqrt{-g(x_n)}} \frac{\delta}{\delta \lambda^{I_1}(x_1)} \cdots \frac{\delta}{\delta \lambda^{I_n}(x_n)} \mathcal{W}[\mathcal{J}] \Big|_{\mathcal{J}=0} \quad (1.3.3)$$

where $\mathcal{J} = 0$ means $(g^{\mu\nu} = \eta^{\mu\nu}, \lambda^I(x) = \lambda^I(\mu) = \text{const}, A_\mu^a = 0)$. Let us insist once more that what we are studying are the effects of *marginal* deformations of a CFT, which is illustrated by fact that in the end we take A_μ^A to be zero (and also m^a).

The assumptions stay the same as before. We assume that diffeomorphisms are anomaly free. In addition, we also assume that the flavor symmetry is anomaly free and that the theory is parity invariant. For a discussion of parity breaking theories with flavor anomalies, see [15]. Weyl symmetry is given by:

$$g^{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g^{\mu\nu}(x) \quad \& \quad \delta_\sigma g^{\mu\nu}(x) = 2\sigma(x) g^{\mu\nu}(x) \quad (1.3.4)$$

The origin of the anomaly has already been discussed (see end of section 1.2.1.1).

1.3.1.2 Weyl symmetry and types of scale transformations

As we have already mentioned, there is nothing conceptually new in the 4 dimensional case compared to the two dimensional one. There are only some additions (at least at this point of the discussion). So this section is essentially a summary sections 1.2.1.2 and 1.2.1.4 adapted to four dimensions.

In four dimensions, the Weyl generator now takes the form:

$$\int d^4x \delta_\sigma \mathcal{J} \frac{\delta}{\delta \mathcal{J}} \equiv \Delta_\sigma = \Delta_\sigma^g - \Delta_\sigma^\beta \quad (1.3.5)$$

where

$$\Delta_\sigma^g = \int d^4x 2\sigma(x) g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} \quad (1.3.6)$$

$$\Delta_\sigma^\beta = - \int d^4x \left(\delta_\sigma \lambda(x) \cdot \frac{\delta}{\delta \lambda(x)} + \delta_\sigma A_\mu^A \frac{\delta}{\delta A_\mu^A(x)} \right) \quad (1.3.7)$$

The Weyl variation is then taken to be of the most general respecting both dimensional analysis and diffeomorphism invariance, which yields:

$$\delta_\sigma \lambda^I(x) = -\sigma(x) \beta^I(\lambda(x)) \quad (1.3.8)$$

$$\delta_\sigma A_\mu^A(x) = -\sigma \rho_I^A \nabla_\mu \lambda^I + S^A \nabla_\mu \sigma \quad (1.3.9)$$

where ∇ denotes the covariant derivative with respect to flavor and geometry. We also have that by dimensional analysis the various coefficients have to be a function of the couplings λ^I . Finally, since Weyl and flavor symmetries commute, the coefficients should be covariant functions.

The local Callan-Symanzik equation then takes the same form as before:

$$\Delta_\sigma \mathcal{W} = \int d^4x \mathcal{A}_\sigma \quad (1.3.10)$$

As before, we can relate this generators to better known symmetries. The transformation of dimensionful parameters:

$$\Delta^\mu \mathcal{W} \equiv \left[\mu \frac{\partial}{\partial \mu} + \int d^4x 2g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} \right] \mathcal{W} = 0 \quad (1.3.11)$$

Notice that had we considered also dimension two operators, there an additional term related to m^a . These can then be combined into an RG generator:

$$\Delta^{RG} \equiv \Delta^\mu + \Delta_{\sigma=-1} \quad (1.3.12)$$

The other generators then take the form:

$$\Delta_\xi^{\text{diff}} = \int d^4x (\nabla_\rho \xi^\mu g^{\rho\nu} + \nabla_\rho \xi^\nu g^{\mu\rho}) \frac{\delta}{\delta g^{\mu\nu}} - \nabla_\mu \xi^\nu A_\nu^A \frac{\delta}{\delta A_\mu^A} - \xi^\rho \left(\nabla_\rho \lambda^I \frac{\delta}{\delta \lambda^I} + \nabla_\rho A_\nu^A \frac{\delta}{\delta A_\nu^A} \right) \quad (1.3.13)$$

$$\begin{aligned} \Delta_c^D &\equiv \Delta_{cx}^{\text{diff}} + \Delta_{-c} \\ &= c \int d^4x \beta^I \frac{\delta}{\delta \lambda^I} + \left(\rho_I^A \nabla_\mu \lambda^I - A_\mu^A \right) \frac{\delta}{\delta A_\mu^A} - x^\rho \left(\nabla_\rho \lambda^I \frac{\delta}{\delta \lambda^I} + \nabla_\rho A_\nu^A \frac{\delta}{\delta A_\nu^A} \right) \end{aligned} \quad (1.3.14)$$

$$\begin{aligned} \Delta_b^K &\equiv \Delta_{2(b \cdot x)x^\mu - x^2 b^\mu}^{\text{diff}} + \Delta_{-2b \cdot x} \\ &= 2b_\mu \int d^4x \left(x^\mu \beta^I \frac{\delta}{\delta \lambda^I} + \left(x^\mu (\rho_I^A \nabla_\mu \lambda^I - A_\mu^A) - \delta_\nu^\mu S^A \right) \frac{\delta}{\delta A_\nu^A} \right) \\ &\quad - \int d^4x (2(b \cdot x)x^\mu - x^2 b^\mu) \left(\nabla_\mu \lambda^I \frac{\delta}{\delta \lambda^I} + \nabla_\rho A_\nu^A \frac{\delta}{\delta A_\nu^A} \right) \end{aligned} \quad (1.3.15)$$

The specific form of these generators is interesting, because it teaches us something about scale versus conformal invariance. Indeed, once we set the source to “zero”, we see that if β^I is equal to zero the theory is going to be scale invariant. On the other hand, we also see that this condition is not sufficient to ensure conformal invariance. Concretely, if we have a theory for which $\beta^I = 0$ but $S^A \neq 0$, the latter will scale but not conformal invariant.

1.3.1.3 Ambiguities and reparametrizations

As we have stated multiple times, one the goals is to express T in a basis of composite operators. For now, this basis also comprises $\nabla_\mu J_A^\mu$ in which case the basis is redundant. The intuitive reason is simple to understand: Since we have the G_F symmetry, the equations of motion will yield a relationship between the different operators in the basis. More concretely, in our way of writing things, the generator of the flavor symmetry takes the form:

$$\Delta_\alpha^F \mathcal{W} \equiv \int d^4x \left[\alpha^A (T_A \lambda)^I \frac{\delta}{\delta \lambda^I(x)} - \nabla_\mu \alpha^A \frac{\delta}{\delta A_\mu^A(x)} \right] \mathcal{W} = 0 \quad (1.3.16)$$

This introduces a possible way of how to modify the Weyl generator. Let us parametrize this definition by taking

$$\alpha^A = -\sigma \omega^A(\lambda) \quad (1.3.17)$$

where the only condition on ω is that it has to be chosen to be covariant. We then make the redefinition:

$$\Delta'_\sigma \equiv \Delta_\sigma + \Delta_{-\sigma\omega}^F \quad (1.3.18)$$

This redefinition is sensible since $\Delta_\alpha^F \mathcal{W} = 0$ and the new generator still commutes with G_F . In terms of the coefficients in the Weyl generator, it corresponds to the following transformation:

$$\beta^I \rightarrow \beta^I + (\omega^A T_A \lambda)^I \quad \& \quad S^A \rightarrow S^A + \omega^A \quad \& \quad \rho_I^A \rightarrow \rho_I^A - \partial_I \omega^A \quad (1.3.19)$$

This very clearly illustrates the fact that there is an ambiguity in the definition of the β function.

This liberty in the definition of Δ_σ is similar to a gauge redundancy, which can be eliminated by “fixing a gauge”, e.g. choosing $\omega^A = -S^A$ so that the new S^A vanishes. The “gauge” invariants are then following quantities:

$$B^I = \beta^I - (S^A T_A \lambda)^I \quad (1.3.20)$$

$$P_I^A = \rho_I^A + \partial_I S^A \quad (1.3.21)$$

In this new basis, the operator equation for T then takes the very simple form:

$$T = B^I \mathcal{O}_I \quad (1.3.22)$$

This may seem like a technicality and while that may be, it is a crucial technicality in the discussion of scale versus conformal invariance. Indeed, the condition for conformal invariance is $B = 0$ and not $\beta = 0$. There exists an example where not using this technicality resulted in wrong conclusions. In their paper [16], the authors believed to have found a perturbative unitary scale invariant theory which was not conformal invariant. The theory was later proven to exactly conformal and the resolution is related to the distinction between β and B .

One can then also define an analog of anomalous dimensions of the composite operators:

$$\gamma_J^I = \partial_I B^J + P_I^A (T_A \lambda) J \quad (1.3.23)$$

$$\gamma_A^B = P_K^B (T_A \lambda)^K \quad (1.3.24)$$

Most the variations of the different quantities which will appear can then partially be written in terms of Lie derivative:

$$\mathcal{L}[Y_{IA\dots}^{JB\dots}] = B^K \partial_K Y_{IA\dots}^{JB\dots} + \gamma_I^K Y_{KA\dots}^{JB\dots} + \gamma_A^C Y_{IC\dots}^{JB\dots} - \gamma_K^J Y_{IA\dots}^{KB\dots} - \gamma_C^B Y_{IA\dots}^{JC\dots} + \dots \quad (1.3.25)$$

In addition, there is also a freedom to reparametrize the sources, which in essence leads to a change in the definition of the renormalized composite operators. It is this freedom of reparametrization which corresponds to the possibility of “improving” an EM tensor. For a more complete discussion, see the corresponding section in [7].

1.3.1.4 Weyl anomaly and consistency conditions

Just as in the two dimensional case, we then have to impose consistency consistency conditions. The first condition, which we impose is that, the Weyl generators have to respect the Abelian nature of Weyl symmetry:

$$[\Delta_{\sigma_1}, \Delta_{\sigma_2}] = 0 \quad (1.3.26)$$

This consistency condition leads to a crucial orthogonality condition:

$$B^I P_I^A = 0 \quad (1.3.27)$$

Next, we have to write down the most general form that the anomaly can take which is given by figure 1.2. Appendix A.3 contains a discussion of the CFT Weyl anomaly.

$$\begin{aligned} \frac{1}{\sqrt{-g}} \mathcal{A}_\sigma = & \sigma \left(\beta_a W^2 + \beta_b E_4 + \frac{1}{9} \beta_c R^2 \right) - \nabla^2 \sigma \left(\frac{1}{3} dR \right) \\ & + \sigma \left(\frac{1}{3} \chi_I^e \nabla_\mu \lambda^I \nabla^\mu R + \frac{1}{6} \chi_{IJ}^f \nabla_\mu \lambda^I \nabla^\mu \lambda^J R + \frac{1}{2} \chi_{IJ}^g G^{\mu\nu} \nabla_\mu \lambda^I \nabla_\nu \lambda^J \right. \\ & \quad \left. + \frac{1}{2} \chi_{IJ}^a \nabla^2 \lambda^I \nabla^2 \lambda^J + \frac{1}{2} \chi_{IJK}^b \nabla_\mu \lambda^I \nabla^\mu \lambda^J \nabla^2 \lambda^K + \frac{1}{4} \chi_{IJKL}^c \nabla_\mu \lambda^I \nabla^\mu \lambda^J \nabla_\nu \lambda^K \nabla^\nu \lambda^L \right. \\ & \quad \left. + \nabla^\mu \sigma \left(G_{\mu\nu} w_I \nabla^\nu \lambda^I + \frac{1}{3} R Y_I \nabla_\mu \lambda^I + \tilde{S}_{IJ} \nabla_\mu \lambda^I (U^{-1})_K^J \nabla^2 \lambda^K + \frac{1}{2} T_{IJK} \nabla_\nu \lambda^I \nabla^\nu \lambda^J \nabla_\mu \lambda^K \right) \right. \\ & \quad \left. - \nabla^2 \sigma \left(U_I \nabla^2 \lambda^I + \frac{1}{2} V_{IJ} \nabla_\nu \lambda^I \nabla^\nu \lambda^J \right) \right. \\ & \quad \left. + \sigma \left(\frac{1}{4} \kappa_{AB} F_{\mu\nu}^A F^{B\mu\nu} + \frac{1}{2} \zeta_{AIJ} F_{\mu\nu}^A \nabla^\mu \lambda^I \nabla^\nu \lambda^J \right) + \nabla^\mu \sigma \left(\eta_{AI} F_{\mu\nu}^A \nabla^\nu \lambda^I \right) \end{aligned}$$

Figure 1.2: Most general form of the anomaly in 4 dimensions. (Taken from [7])

One then has to impose the WZ consistency conditions on the anomaly:

$$\Delta_{\sigma_1} \int d^4 x \mathcal{A}_{\sigma_2} = \Delta_{\sigma_2} \int d^4 x \mathcal{A}_{\sigma_1} \quad (1.3.28)$$

This then leads to a number of differential equation relating the various tensors appearing in the anomaly. The result which was obtained in [7] is that by using the freedom to add

a local functional of the sources to the generating functional one can eliminate a “suitable set of scheme dependent terms”, such that the consistency conditions become algebraic equations (rather than differential). In the end, the result which will be of most interest to us is that one of these algebraic equations is the four dimensional equivalent of what we have obtained in two dimensions:

$$\mathcal{L}[w_I] = -8\partial_I a + \chi_{IJ}^g B^J \quad (1.3.29)$$

There are also two other consistency conditions which read:

$$\mathcal{L}[\eta_{AI}] = \kappa_{AB} P_I^B + \zeta_{AIJ} B^J - \chi_{IJ}^g (T_A \lambda)^J \quad (1.3.30)$$

$$0 = \eta_{AI} B^I + w_I (T_A \lambda)^I \quad (1.3.31)$$

Let us now define the new quantity:

$$\tilde{a} = a + \frac{1}{8} w_I B^I \quad (1.3.32)$$

* * * * *

We then compute:

$$\begin{aligned} 8\partial_I \tilde{a} &= 8\partial_I a + (\partial_I w_J) B^J + w_J \partial_I B^J \\ &= \chi_{IJ}^g B^J - \mathcal{L}[w_I] + B^J \partial_I w_J + w_J \partial_I B^J \\ &= \chi_{IJ}^g B^J - B^K \partial_K w_I - \gamma_I^K w_K + B^J \partial_I w_J + w_J \partial_I B^J \\ &= (\chi_{IJ}^g + \partial_I w_J - \partial_J w_I) B^J + w_J \partial_I B^J - \gamma_I^K w_K \\ &= (\chi_{IJ}^g + \partial_I w_J - \partial_J w_I) B^J - P_I^A (T_A \lambda)^J w_J \\ &= (\chi_{IJ}^g + \partial_I w_J - \partial_J w_I) B^J + P_I^A \eta_{AJ} B^J \end{aligned} \quad (1.3.33)$$

where in the third line we used the definition of the Lie derivative:

$$\mathcal{L}[Y_{IA...}^{JB...}] = B^K \partial_K Y_{IA...}^{JB...} + \gamma_I^K Y_{KA...}^{JB...} + \gamma_A^C Y_{IC...}^{JB...} - \gamma_K^J Y_{IA...}^{KB...} - \gamma_C^B Y_{IA...}^{JC...} + \dots \quad (1.3.34)$$

In the fifth line we used the definition of the anomalous dimension matrix:

$$\gamma_J^I = \partial_I B^J + P_I^A (T_A \lambda)^J \quad (1.3.35)$$

and in the last line, we used the consistency condition:

$$0 = \eta_{AI} B^I + w_I (T_A \lambda)^I \quad (1.3.36)$$

* * * * *

A short calculation then yields a gradient flow formula in four dimensions:

$$8\partial_I \tilde{a} = (\chi_{IJ}^g + \partial_I w_J - \partial_J w_I + P_I^A \eta_{AJ}) B^J \quad (1.3.37)$$

This formula has several interesting implications: It “gives non-trivial relations among the perturbative expansion coefficients of the β -function” and of the other quantities in the right hand side. It also implies that $\tilde{a}$ is stationary at a conformal fixed point, since we have already pointed out that such a fixed point is characterized by $B^I = 0$. The most interesting consequence comes from contracting that relation with B which yields:

$$8\mu \frac{d}{d\mu} \tilde{a} \equiv 8B^I \partial_I \tilde{a} = \chi_{IJ}^g B^I B^J \quad (1.3.38)$$

Notice that here again, the orthogonality condition between P and B plays a crucial role. We are going to discuss the implications of this result later on once we have proven (in perturbation theory) that χ_{IJ}^g is positive definite.

1.3.2 Off-criticality correlators and the dilaton amplitude

1.3.2.1 Dilaton effective action

The motivation for introducing a dilaton effective action and the procedure of doing so are exactly the same as in section 1.2.1.5 except for the exponential growth of messy computations. We will therefore only report the results.

The first interesting result is that the unique solution for $A_\mu^A[\tau, y]$ is the solution where $A_\mu^A[\tau, y]$ is identically zero. This result depends on the “gauge” choice we made earlier on, taking $S^A = 0$, which was equivalent to eliminating the currents in the expansion of T . The new result then implies that currents can be eliminated completely from Ward identities even in the contact terms.

The main contributions to Γ_{local} are:

$$\Gamma^{W^2}[g, \tau] = - \int d^4x \sqrt{-g} C(\lambda(\mu), \tau) W^2[g] \quad (1.3.39)$$

$$\begin{aligned} \Gamma^{E_4}[g, \tau] = & \int d^4x \sqrt{-g} \left(A(\lambda(\mu), \tau) E_4[g] \right. \\ & + \tilde{a}(\lambda(\mu e^\tau)) \left(4G^{\mu\nu}[g] \nabla_\mu \nabla_\nu \tau - 4\nabla^2 \tau (\nabla \tau)^2 + 2(\nabla \tau)^4 \right) \\ & \left. - \mathcal{L}[\tilde{a}](\lambda(\mu e^\tau)) (\nabla \tau)^4 + \dots \right) \end{aligned} \quad (1.3.40)$$

where

$$E_4 = R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \quad (1.3.41)$$

is the Euler density,

$$W_{\mu\nu\lambda\rho} W^{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \quad (1.3.42)$$

is the Weyl tensor squared and

$$C(\lambda(\mu), \tau) = \int_\mu^{\mu e^\tau} c(\lambda(\bar{\mu})) d \ln \bar{\mu} \quad \& \quad A(\lambda(\mu), \tau) = \int_\mu^{\mu e^\tau} a(\lambda(\bar{\mu})) d \ln \bar{\mu} \quad (1.3.43)$$

The dots stand for terms of order B^2 , which additionally vanish on-shell, where the on-shell condition is defined as

$$\nabla^2 e^{-\tau} = 0 \quad \Leftrightarrow \quad \nabla^2 \phi = 0 \quad (1.3.44)$$

where ϕ is the canonical dilaton defined by $e^{-\tau} = 1 + \phi$. We will discuss why this is a relevant and absolutely crucial condition in section 1.3.2.2. The point of all of this is that since

$$8\mathcal{L}[\tilde{a}] = \chi_{IJ}^g B^I B^J \rightarrow O(B^2) \quad (1.3.45)$$

we have proven that, close to a fixed point, where B is small parameter, the local part of the on-shell dilaton effective action reduces to:

$$\Gamma_{\text{local}}[\eta, \tau] = \tilde{a} \int d^4x \left(-4\nabla^2 \tau (\nabla \tau)^2 + 2(\nabla \tau)^4 \right) + O(B^2) \quad (1.3.46)$$

which again is exactly what we expected since it corresponds to WZ dilaton action in flat space given in equation (1.2.106). The highly non-trivial result however is that any corrections begin at order B^2 .

The result for the non-local part of the dilaton action is exactly the same as in two dimensions:

$$\mathcal{W}[\lambda[\tau, 1]] =: \exp \left\{ \int d^4x (\lambda[\tau, 1] - \lambda^I(\mu)) \frac{\delta}{\delta \bar{\lambda}^I(x)} \right\} : \mathcal{W}[\bar{\lambda}] \Big|_{\bar{\lambda}=\lambda(\mu)} \quad (1.3.47)$$

which to second order in ϕ gives

$$\begin{aligned} \Gamma_{\text{non-local}}[\eta, \tau] &= \mathcal{W}[\lambda(\mu)] - \int d^4x \phi(x) B^I \frac{\delta}{\delta \bar{\lambda}^I(x)} \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \\ &+ \int d^4x \int d^4y \frac{\phi(y)\phi(x)}{2} \left[\delta(x-y) B^J \left(\delta_J^I + \partial_J B^I \right) \frac{\delta}{\delta \bar{\lambda}^I(x)} + B^I B^J \frac{\delta}{\delta \bar{\lambda}^I(x)} \frac{\delta}{\delta \bar{\lambda}^J(y)} \right] \mathcal{W} \Big|_{\bar{\lambda}=\lambda(\mu)} \end{aligned} \quad (1.3.48)$$

At this point, we want to make a little aside on the dimension 2 operators. Recall from section 1.2.1.5 that to compute the dilaton effective action, we needed to find the Weyl transformed interpolating sources. We already mentioned that for the background gauge fields, the answer was very simple and convenient, namely that A_μ^A simply vanishes (for a specific choice of gauge). For the dimension 2 operators, things are in general less convenient, except in the on-shell case where things get again pretty simple. Indeed, all contributions to the local part of the dilaton effective action vanish on-shell and there also exists a specific scheme in which the contributions to the non-local part of the dilaton effective action also vanish. This was the original purpose of [7]: To prove that the possible contribution of $\square \mathcal{O}_a$ to the correlators of T do not invalidate any of the results.

1.3.2.2 Dilaton amplitude

As in the two dimensional case, the quantity we will be interested in, is the dilaton “scattering” amplitude, which in four dimensions is defined by:

$$(2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) A(p_1, p_2, p_3, p_4) = \mathcal{F} \left\{ \frac{\delta}{\delta \phi(x_1)} \cdots \frac{\delta}{\delta \phi(x_4)} \mathcal{W}[g, \phi] \Big|_{\substack{g=\eta \\ \phi=0}} \right\} (p_1, \dots, p_4) \quad (1.3.49)$$

* * * * *

Let us now relate A to the correlators of T . To do so, notice that:

$$\frac{\delta}{\delta\phi(x)} = \int d^4y \frac{\delta\tau(y)}{\delta\phi(x)} \frac{\delta}{\delta\tau(y)} = -(1+\phi)^{-1} \frac{\delta}{\delta\tau(x)} = -e^\tau \frac{\delta}{\delta\tau(x)} \quad (1.3.50)$$

To make the following calculation a bit more palatable, we introduce some new notation: If a $\delta\ldots$ is situated just before $\mathcal{W}$, then it represents a functional derivative, if not it simply represents a delta function, e.g. $\delta_{24} \equiv \delta(x_2 - x_4)$. We then have:

$$\begin{aligned} \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_4)} \mathcal{W}[g, \phi] &= e^{\tau_1} \delta_1 e^{\tau_2} \delta_2 e^{\tau_3} \delta_3 e^{\tau_4} \delta_4 \mathcal{W} \\ &= e^{\tau_1} \delta_1 e^{\tau_2} \delta_2 e^{\tau_3} e^{\tau_4} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) \\ &= e^{\tau_1} \delta_1 e^{\tau_2} e^{\tau_3} e^{\tau_4} (\delta_{23} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + \delta_{24} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + (\delta_{34} \delta_{24} \mathcal{W} + \delta_{234} \mathcal{W})) \\ &= e^{\tau_1} e^{\tau_2} e^{\tau_3} e^{\tau_4} (\delta_{12} (\delta_{23} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + \delta_{24} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + (\delta_{34} \delta_{24} \mathcal{W} + \delta_{234} \mathcal{W})) \\ &\quad + \delta_{13} (\delta_{23} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + \delta_{24} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + (\delta_{34} \delta_{24} \mathcal{W} + \delta_{234} \mathcal{W})) \\ &\quad + \delta_{14} (\delta_{23} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + \delta_{24} (\delta_{34} \delta_4 \mathcal{W} + \delta_{34} \mathcal{W}) + (\delta_{34} \delta_{24} \mathcal{W} + \delta_{234} \mathcal{W})) \\ &\quad + (\delta_{23} (\delta_{34} \delta_{14} \mathcal{W} + \delta_{134} \mathcal{W}) + \delta_{24} (\delta_{34} \delta_{14} \mathcal{W} + \delta_{134} \mathcal{W}) + (\delta_{34} \delta_{124} \mathcal{W} + \delta_{1234} \mathcal{W}))) \\ &= e^{\tau_1} e^{\tau_2} e^{\tau_3} e^{\tau_4} [\delta_4 \mathcal{W} (\delta_{12} \delta_{23} \delta_{34} + \delta_{12} \delta_{24} \delta_{34} + \delta_{13} \delta_{23} \delta_{34} + \delta_{13} \delta_{24} \delta_{34} + \delta_{14} \delta_{23} \delta_{34} + \delta_{14} \delta_{24} \delta_{34}) \\ &\quad + \delta_{34} \mathcal{W} (\delta_{12} \delta_{23} + \delta_{12} \delta_{24} + \delta_{13} \delta_{23} + \delta_{13} \delta_{24} + \delta_{14} \delta_{23} + \delta_{14} \delta_{24}) \\ &\quad + \delta_{24} \mathcal{W} (\delta_{12} \delta_{34} + \delta_{13} \delta_{34} + \delta_{14} \delta_{34}) \\ &\quad + \delta_{14} \mathcal{W} (\delta_{23} \delta_{34} + \delta_{24} \delta_{34}) \\ &\quad + \delta_{234} \mathcal{W} (\delta_{12} + \delta_{13} + \delta_{14}) \\ &\quad + \delta_{134} \mathcal{W} (\delta_{23} + \delta_{24}) + \delta_{124} \mathcal{W} \delta_{34} + \delta_{1234} \mathcal{W}] \end{aligned}$$

Now, using the fact that:

$$\langle T(x_1) \cdots T(x_n) \rangle = \frac{(-i)^{n-1}}{\sqrt{-g(x_1)} \cdots \sqrt{-g(x_n)}} \frac{\delta}{\delta\tau(x_1)} \cdots \frac{\delta}{\delta\tau(x_n)} \mathcal{W} \quad (1.3.51)$$

and evaluating at $g = \eta$ and $\phi = 0$, we obtain:

$$\begin{aligned} \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_4)} \mathcal{W}[g, \phi] \Big|_{\substack{g=\eta \\ \phi=0}} &= \langle T(x_4) \rangle (\delta_{12} \delta_{23} \delta_{34} + \delta_{12} \delta_{24} \delta_{34} + \delta_{13} \delta_{23} \delta_{34} + \delta_{13} \delta_{24} \delta_{34} + \delta_{14} \delta_{23} \delta_{34} + \delta_{14} \delta_{24} \delta_{34}) \\ &\quad + (-i) \langle T(x_3) T(x_4) \rangle (\delta_{12} \delta_{23} + \delta_{12} \delta_{24} + \delta_{13} \delta_{23} + \delta_{13} \delta_{24} + \delta_{14} \delta_{23} + \delta_{14} \delta_{24}) \\ &\quad + i \langle T(x_2) T(x_4) \rangle (\delta_{12} \delta_{34} + \delta_{13} \delta_{34} + \delta_{14} \delta_{34}) \\ &\quad + i \langle T(x_1) T(x_4) \rangle (\delta_{23} \delta_{34} + \delta_{24} \delta_{34}) \\ &\quad + (-1) \langle T(x_2) T(x_3) T(x_4) \rangle (\delta_{12} + \delta_{13} + \delta_{14}) \\ &\quad + (-1) \langle T(x_1) T(x_3) T(x_4) \rangle (\delta_{23} + \delta_{24}) + (-1) \langle T(x_1) T(x_2) T(x_4) \rangle \delta_{34} + (-i) \langle T(x_1) T(x_2) T(x_3) T(x_4) \rangle \end{aligned}$$

We now want to take the Fourier transform. Let us look at two examples to see what happens:

$$\begin{aligned} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \langle T_2 T_3 T_4 \rangle \delta_{13} e^{-ip_j x^j} &= \int d^4x_2 d^4x_3 d^4x_4 \langle T_2 T_3 T_4 \rangle e^{-i(p_2 x_2 + (p_1 + p_3)x_3 + p_4 x_4)} \\ &= \langle T(p_1 + p_3) T(p_2) T(p_4) \rangle \end{aligned}$$

$$\begin{aligned} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \langle T_3 T_4 \rangle \delta_{12} \delta_{23} e^{-ip_j x^j} &= \int d^4x_3 d^4x_4 \langle T_3 T_4 \rangle e^{-i((p_1 + p_2 + p_3)x_3 + p_4 x_4)} \\ &= \langle T(p_1 + p_2 + p_3) T(p_4) \rangle \end{aligned}$$

$$\begin{aligned} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \langle T_3 T_4 \rangle \delta_{13} \delta_{24} e^{-ip_j x^j} &= \int d^4x_3 d^4x_4 \langle T_3 T_4 \rangle e^{-i((p_1+p_3)x_3 + (p_2+p_4)x_4)} \\ &= \langle T(p_1 + p_3) T(p_2 + p_4) \rangle \end{aligned}$$

So that we end up with:

$$\begin{aligned} \mathcal{F}\left\{\frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_4)} \mathcal{W}[g, \phi] \Big|_{\substack{g=\eta \\ \phi=0}}\right\}(p_1, \dots, p_4) \\ &= i(\langle T(p_1 + p_2 + p_3) T(p_4) \rangle + \langle T(p_1 + p_2 + p_4) T(p_3) \rangle + \langle T(p_1 + p_2 + p_3) T(p_4) \rangle \\ &\quad + \langle T(p_1 + p_3) T(p_2 + p_4) \rangle + \langle T(p_1 + p_4) T(p_2 + p_3) \rangle + \langle T(p_1 + p_2 + p_4) T(p_3) \rangle) \\ &\quad + i(\langle T(p_1 + p_2) T(p_3 + p_4) \rangle + \langle T(p_1 + p_3 + p_4) T(p_2) \rangle + \langle T(p_1 + p_3 + p_4) T(p_2) \rangle \\ &\quad + \langle T(p_2 + p_3 + p_4) T(p_1) \rangle + \langle T(p_2 + p_3 + p_4) T(p_1) \rangle) \\ &\quad - (\langle T(p_1 + p_2) T(p_3) T(p_4) \rangle + \langle T(p_1 + p_3) T(p_2) T(p_4) \rangle + \langle T(p_1 + p_4) T(p_2) T(p_3) \rangle \\ &\quad + \langle T(p_2 + p_3) T(p_1) T(p_4) \rangle + \langle T(p_2 + p_4) T(p_1) T(p_3) \rangle + \langle T(p_3 + p_4) T(p_1) T(p_2) \rangle) \\ &\quad + (-i) \langle T(p_1) T(p_2) T(p_3) T(p_4) \rangle \\ &= 2i(\langle T(p_1 + p_2 + p_3) T(p_4) \rangle + \langle T(p_1 + p_2 + p_4) T(p_3) \rangle + \langle T(p_1 + p_3 + p_4) T(p_2) \rangle + \langle T(p_2 + p_3 + p_4) T(p_1) \rangle) \\ &\quad + i(\langle T(p_1 + p_3) T(p_2 + p_4) \rangle + \langle T(p_1 + p_4) T(p_2 + p_3) \rangle + \langle T(p_1 + p_2) T(p_3 + p_4) \rangle) \\ &\quad - (\langle T(p_1 + p_2) T(p_3) T(p_4) \rangle + \langle T(p_1 + p_3) T(p_2) T(p_4) \rangle + \langle T(p_1 + p_4) T(p_2) T(p_3) \rangle \\ &\quad + \langle T(p_2 + p_3) T(p_1) T(p_4) \rangle + \langle T(p_2 + p_4) T(p_1) T(p_3) \rangle + \langle T(p_3 + p_4) T(p_1) T(p_2) \rangle) \\ &\quad + (-i) \langle T(p_1) T(p_2) T(p_3) T(p_4) \rangle \end{aligned}$$

To summarize the result, we have:

$$\begin{aligned} (2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) A(p_1, p_2, p_3, p_4) &= -i \langle T(p_1) T(p_2) T(p_3) T(p_4) \rangle \\ &\quad - (\langle T(p_1 + p_2) T(p_3) T(p_4) \rangle + \text{permutations}) \\ &\quad + i(\langle T(p_1 + p_3) T(p_2 + p_4) \rangle + \text{permutations}) \\ &\quad + 2i(\langle T(p_1 + p_2 + p_3) T(p_4) \rangle + \text{permutations}) \end{aligned}$$

In the forward region where $t = 0$, we have:

$$\begin{aligned} (2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) A(p_1, p_2, p_3, p_4) \Big|_{t=0} &= \\ &2i(\langle T(p_2) T(-p_2) \rangle + \langle T(p_1) T(-p_1) \rangle + \langle T(p_2) T(-p_2) \rangle + \langle T(p_1) T(-p_1) \rangle) \\ &\quad + i(\langle T(0) T(0) \rangle + \langle T(p_1 - p_2) T(-(p_1 - p_2)) \rangle + \langle T(p_1 + p_2) T(-(p_1 + p_2)) \rangle) \\ &\quad - (\langle T(p_1 + p_2) T(-p_1) T(-p_2) \rangle + \langle T(0) T(p_2) T(-p_2) \rangle + \langle T(p_1 - p_2) T(p_2) T(-p_1) \rangle \\ &\quad + \langle T(p_2 - p_1) T(p_1) T(-p_2) \rangle + \langle T(0) T(p_1) T(-p_1) \rangle + \langle T(-(p_1 + p_2)) T(p_1) T(p_2) \rangle) \\ &\quad + (-i) \langle T(p_1) T(p_2) T(-p_1) T(-p_2) \rangle \\ \Rightarrow &= 4i(\langle T(p_2) T(-p_2) \rangle + \langle T(p_1) T(-p_1) \rangle) \\ &\quad + i(\langle T(0) T(0) \rangle + \langle T(p_1 - p_2) T(-(p_1 - p_2)) \rangle + \langle T(p_1 + p_2) T(-(p_1 + p_2)) \rangle) \\ &\quad - (\langle T(p_1 + p_2) T(-p_1) T(-p_2) \rangle + \langle T(-(p_1 + p_2)) T(p_1) T(p_2) \rangle) \\ &\quad - (\langle T(0) T(p_2) T(-p_2) \rangle + \langle T(0) T(p_1) T(-p_1) \rangle) \\ &\quad - (\langle T(p_2 - p_1) T(p_1) T(-p_2) \rangle + \langle T(p_1 - p_2) T(p_2) T(-p_1) \rangle) \\ &\quad + (-i) \langle T(p_1) T(p_2) T(-p_1) T(-p_2) \rangle \end{aligned}$$

Therefore, if we now furthermore impose the on-shell condition $p_i^2 = 0$ and the forward limit $t = 0$ with the same arguments of dimensional analysis, the above expression reduces

to:

$$(2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) A(p_1, p_2, p_3, p_4) \Big|_{t=0} = +i \langle T(p_1 + p_2) T(-(p_1 + p_2)) \rangle \\ - \langle T(p_1 + p_2) T(-p_1) T(-p_2) \rangle - \langle T(-(p_1 + p_2)) T(p_1) T(p_2) \rangle + (-i) \langle T(p_1) T(p_2) T(-p_1) T(-p_2) \rangle$$

* * * * *

It is interesting to see how the DSA is related to correlators of T , because it gives an indication as to why the four dimensional case is so much harder than the two dimensional one. Indeed, the above separated section shows that the DSA can be written as:

$$(2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) A(p_1, p_2, p_3, p_4) = -i \langle T(p_1) T(p_2) T(p_3) T(p_4) \rangle \\ - (\langle T(p_1 + p_2) T(p_3) T(p_4) \rangle + \text{permutations}) \\ + i (\langle T(p_1 + p_3) T(p_2 + p_4) \rangle + \text{permutations}) \\ + 2i (\langle T(p_1 + p_2 + p_3) T(p_4) \rangle + \text{permutations})$$

which is a non-trivial linear combination of 2-, 3- and 4-point functions.

Again, for generic dynamics, this amplitude is UV divergent and therefore requires renormalization. The question is, as it was in two dimensions, whether there exists a condition which permits us to circumvent that problem, i.e. a condition which would imply the finiteness of the amplitude? There exists such a condition and it can be understood in two different ways, both of which are informative. In our language of the local RG, we have already shown that the UV divergences are related to the integral of the anomaly for constant σ , so the question becomes about whether or not there exists a condition that implies that the integral of the anomaly is zero, so that the amplitude is RG independent. In a pure dilaton background, i.e. ($\lambda^I = \text{const}$, $A_\mu^A = 0$), the only terms of the anomaly that matter are the ones depending only on the metric: E_4 is the Euler density, which vanishes when integrated over a topologically trivial space and the Weyl tensor squared vanishes for conformally flat metrics. So what we need is a condition which ensures that the Ricci scalar squared vanishes. In the pure dilaton background, the Ricci scalar is given by:

$$R[\eta, \tau] = e^{3\tau} \square e^{-\tau} = \frac{1}{(1 + \phi)^3} \square \phi \quad (1.3.52)$$

which therefore vanishes if we impose an on-shell condition for the canonical dilaton, which reads:

$$\square \phi = 0 \quad (1.3.53)$$

As we have already mentioned in the previous section, this condition makes life much simpler in a lot of places in the calculation. The other way of understanding this is that any counterterm you can possibly write down is either the Euler density, the Weyl tensor squared or proportional to the Ricci scalar, which all vanish in a pure *on-shell* dilaton background. Counterterms involving currents or tensors are excluded by unitarity bounds. The impossibility of writing down any counterterms (up to the cosmological constant term) is then equivalent to the finiteness of the amplitude.

From the above discussion, it then follows that the DSA has to take the form:

$$A(s, t) = s^2 F\left(\frac{s}{\mu^2}, \frac{t}{\mu^2}, \lambda(\mu)\right) + \Lambda \quad (1.3.54)$$

1.3.3 Proving the a-theorem(s) and constraining RG flows

1.3.3.1 Non-perturbative weak a-theorem

In this section, we will prove the non-perturbative weak a-theorem. The method is exactly the same as in two dimensions. Therefore the beginning will essentially be a verbatim rewriting of section 1.2.2.2.

We assume that there exists an UV CFT whose deformation by relevant operators generates an RG flow towards another IR CFT fixed point. At the UV fixed point, we can then write:

$$S = S_{UV} + \int d^4x \sqrt{-\hat{g}} \sum_i c_i m^{4-\Delta_i} \hat{\mathcal{O}}_i \quad (1.3.55)$$

where S_{UV} is the action of the UV CFT, $\hat{\mathcal{O}}_i$ relevant and marginal primary operators which therefore have dimension $\Delta_i \leq 4$, m is the mass scale of the associated flow, which for simplicity we assume to be the unique characteristic mass scale controlling the flow and the c_i are dimensionless coefficients. We have to separate two cases: the marginal operators and the strictly relevant operators. We already know what the effect of the marginal operators will be. Indeed, one of the reasons to study the local RG equation is because it answers that specific question. More precisely, we have proven that in the presence of marginal deformations the generating separates into a local and a non-local contribution given in equations (1.3.46) and (1.3.48) and the DSA then takes the form (1.3.54). In the case of the strictly relevant operators, since we are only interested in the leading behavior very close to the fixed point, dimensional analysis is an adequate guide.

Let us now consider the dilaton couplings arising from the relevant deformations. We had written the original action in terms of the fields of the CFT denoted by $\hat{\Phi}$ and the background metric. If we now define (in addition to the definition of $\hat{g}$ given previously):

$$\hat{\Phi} = e^{\tau \Delta_\Phi} \Phi \quad (1.3.56)$$

then the relevant deformations take the form:

$$S_{rel}[\hat{\Phi}, \hat{g}] = \int d^4x \sqrt{-g} (m e^{-\tau})^{4-\Delta} \mathcal{O}[\Phi, g] \quad (1.3.57)$$

where the power of two came from the metric determinant and the power of Δ from the combination of $\hat{\Phi}$ in $\mathcal{O}_i$.

* * * * *

We start by rewriting Γ_{local} in a more convenient form. We have:

$$\square e^{-\tau} = ((\partial\tau)^2 - \square\tau) e^{-\tau} \Rightarrow \square\tau = (\partial\tau)^2 - e^\tau \square e^{-\tau} \quad (1.3.58)$$

The on-shell condition then implies:

$$\square\tau = (\partial\tau)^2 \quad (1.3.59)$$

So that finally:

$$\begin{aligned}
\Gamma_{local}[\eta, \tau] &= \tilde{a} \int d^4x \left(-4\nabla^2 \tau (\nabla \tau)^2 + 2(\nabla \tau)^4 \right) + O(B^2) \\
&= -2\tilde{a} \int d^4x (\nabla \tau)^4 + O(B^2) \\
&= -2\tilde{a} \int d^4x \frac{1}{(1+\phi)^4} (\nabla \phi)^4 + O(B^2)
\end{aligned} \tag{1.3.60}$$

The contribution from the local part of the dilaton effective action to the DSA is then given by:

$$\begin{aligned}
\frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_4)} \Gamma_{local}[\eta, \phi] \Big|_{\phi=0} &= -16\tilde{a} \int d^4x \left(\partial_\mu^x \delta(x-x_1) \partial_x^\mu \delta(x-x_2) \partial_\nu^x \delta(x-x_3) \partial_x^\nu \delta(x-x_4) \right. \\
&\quad + \partial_\mu^x \delta(x-x_1) \partial_x^\mu \delta(x-x_3) \partial_\nu^x \delta(x-x_2) \partial_x^\nu \delta(x-x_4) \\
&\quad \left. + \partial_\mu^x \delta(x-x_1) \partial_x^\mu \delta(x-x_4) \partial_\nu^x \delta(x-x_2) \partial_x^\nu \delta(x-x_3) \right) \\
\Rightarrow A_{local}(s, t) &= -16\tilde{a} ((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)) \\
&= -4\tilde{a} (s^2 + t^2 + (s+t)^2)
\end{aligned} \tag{1.3.61}$$

where used the fact that $(p_1 \cdot p_2) = \frac{s}{2}$, $(p_1 \cdot p_3) = \frac{t}{2}$ and $(p_1 \cdot p_4) = \frac{u}{2}$. The contribution from $\Gamma_{non-local}$ will be of order B^2 and therefore does not matter for this computation. It will actually matter later where we will have more to say about the contribution it has to the imaginary part of the amplitude.

For the contribution of the (ir)relevant deformations, the same type of dimensional analysis arguments than in two dimensions, then yields that they are subleading because of power suppression $\left(\frac{m^2}{s}\right)^\# \left(\frac{s}{m^2}\right)^\#$.

* * * * *

Using the results from the local RG equation, as well as some basic dimensional analysis, it is then easy to show that at the conformal fixed points the DSA reduces to:

$$A(s, 0) \xrightarrow{\frac{s}{m^2} \rightarrow \infty} -8s^2 a_{UV} \left(1 + \left(\frac{m^2}{s} \right)^{4-\Delta_{UV}} \right) + \Lambda_{cc}^{UV} \tag{1.3.62}$$

$$A(s, 0) \xrightarrow{\frac{s}{m^2} \rightarrow 0} -8s^2 a_{IR} \left(1 + \left(\frac{s}{m^2} \right)^{\Delta_{IR}-4} \right) + \Lambda_{cc}^{IR} \tag{1.3.63}$$

To prove the result, we will now use a dispersion relation. Let us define:

$$\eta_m \equiv \frac{s}{m^2} \tag{1.3.64}$$

The Cauchy theorem then implies:

$$\frac{1}{2\pi i} \oint_C \frac{A(\eta_m, 0)}{\eta_m^3} d\eta_m = 0 \tag{1.3.65}$$

where the contour is shown in figure 1.3 (and t is fixed). The inner half-circle of the contour is taken with a radius $\varepsilon \rightarrow 0$.

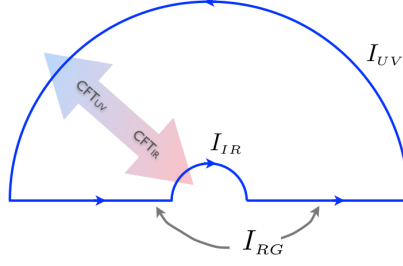


Figure 1.3: Contour for the dispersion relation in four dimensions.

With what we have seen from the two dimensional case, we know that the above integral is going to involve the imaginary part of the DSA and we will want to conclude the a-theorem by using positivity of the latter. In general, positivity of a 2-by-2 scattering amplitude is not guaranteed, but if we go to the forward limit $t = 0$, then the optical theorem relates the imaginary part of the amplitude to the the total cross-section which is positive. This explains why above we have already taken the forward limit $t = 0$.

We can now compute:

$$\begin{aligned}
 I_{UV} &= \frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_0^\pi i d\theta r e^{i\theta} \left(-\frac{8m^2 a_{UV}}{r e^{i\theta}} + \frac{\Lambda_{cc}^{UV}}{r^3 e^{3i\theta}} \right) \\
 &= -8m^2 a_{UV} + \frac{i\Lambda_{cc}^{UV}}{2\pi i} \int_0^\pi e^{-2i\theta} \\
 &= -8m^2 a_{UV}
 \end{aligned} \tag{1.3.66}$$

In exactly the same way, we obtain:

$$I_{IR} = 8m^2 a_{IR} \tag{1.3.67}$$

To compute the last part of the contour, we need again the Schwarz reflection principle as well as another property not needed in two dimensions, namely crossing. Indeed, since we are in the massless forward limit we have $s = -u$ so that crossing symmetry dictates:

$$A(s, 0) = A(u, 0) = A(-s, 0) \tag{1.3.68}$$

So that the last part of the computation yields:

$$\begin{aligned}
 I_{RG} &= \frac{1}{2\pi i} \int_{-\infty}^0 \frac{A(\eta_m + i\delta, 0)}{\eta_m^3} d\eta_m + \frac{1}{2\pi i} \int_0^\infty \frac{A(\eta_m + i\delta, 0)}{\eta_m^3} d\eta_m \\
 &= \frac{1}{2\pi i} \int_0^\infty \frac{A(\eta_m + i\delta, 0) - A(-\eta_m + i\delta, 0)}{\eta_m^3} d\eta_m \\
 &= \frac{1}{2\pi i} \int_0^\infty \frac{A(\eta_m + i\delta, 0) - A(\eta_m - i\delta, 0)}{\eta_m^3} d\eta_m \\
 &= \frac{1}{2\pi i} \int_0^\infty \frac{A(\eta_m + i\delta, 0) - A(\eta_m + i\delta, 0)^*}{\eta_m^3} d\eta_m \\
 &= \frac{1}{\pi} \int_0^\infty \frac{\text{Im} A(\eta_m, 0)}{\eta_m^3} d\eta_m
 \end{aligned} \tag{1.3.69}$$

Finally, we find:

$$a_{UV} - a_{IR} = \frac{1}{8m^2\pi} \int_0^\infty \frac{\text{Im}A(\eta_m, 0)}{\eta_m^3} d\eta_m \quad (1.3.70)$$

$$= \frac{1}{8\pi} \int_0^\infty \frac{\text{Im}A(s, 0)}{s^3} ds \quad (1.3.71)$$

As we already mentioned, unitarity then implies the following positivity constraint:

$$\text{Im}A(s, 0) \geq 0 \quad (1.3.72)$$

So that now we are able to conclude:

$$a_{UV} - a_{IR} \geq 0 \quad \Leftrightarrow \quad a_{UV} \geq a_{IR} \quad (1.3.73)$$

1.3.3.2 Positive definiteness of χ_{IJ}^g and constraints on RG flow asymptotics

In this section, we will prove that the matrix χ_{IJ}^g appearing as a coefficient in the anomaly is positive definite in perturbation theory, which then has several consequences. The steps in this section are nearly exactly the same than in the two-dimensional case.

We parametrize the DSA as:

$$A(s, 0) = s^2 F\left(\frac{s}{\mu^2}, 0, \lambda(\mu)\right) + \Lambda \equiv -8s^2 \alpha(s) + \Lambda \quad (1.3.74)$$

We can now write again a dispersion relation:

$$\frac{1}{2\pi i} \oint_C \frac{A(s, 0)}{s^3} = 0 \quad (1.3.75)$$

where the contour is now the same as in figure 1.3 except for the fact the radii are now taken to be s_1 and s_2 instead of 0 and ∞ . As before, we also define the “average amplitude”:

$$\bar{\alpha}(s) = \frac{1}{\pi} \int_0^\pi \alpha(se^{i\theta}) \quad (1.3.76)$$

which then follows:

$$\bar{\alpha}(s_2) - \bar{\alpha}(s_1) = \frac{2}{\pi} \int_{s_1}^{s_2} \frac{ds}{s} (-\text{Im}\alpha(s)) \geq 0 \quad (1.3.77)$$

Again, we have that by the Schwarz reflection principle $\bar{\alpha}$ is real and since the amplitude is finite, the difference also has to be finite.

Concerning the left hand side, we have already proven in the previous section that:

$$\alpha_{\text{local}}(s) = \tilde{a}(\lambda(\mu)) + O(B^2) \quad \& \quad \alpha_{\text{non-local}}(s) = O(B^2) \quad (1.3.78)$$

Notice that both parts of α are independent of s , so that we finally obtain that:

$$\bar{\alpha}(s) = \tilde{a}(\lambda(\mu)) + O(B^2) = \tilde{a}(\lambda(\sqrt{s})) + O(B^2) \quad (1.3.79)$$

where in the second equality we used the fact that the amplitude is μ independent. It is at this point that the liberty of adding local functionals of the sources to the generating functional, which essentially corresponds to the liberty of choosing a different renormalization scheme, plays a crucial role. Indeed, we can now add to $\mathcal{W}$ the local term

$$\frac{c_{IJ}}{2} \sqrt{g} G^{\mu\nu} \nabla_\mu \lambda^I \nabla_\nu \lambda^J \quad (1.3.80)$$

which trivially does not affect the DSA since the latter is computed at $\lambda = \text{const}$ but generates the following transformation of $\tilde{a}$ and χ_{IJ}^g :

$$\tilde{a} \rightarrow \tilde{a} + B^I B^J c_{IJ} \quad \& \quad \chi_{IJ}^g \rightarrow \chi_{IJ}^g + \mathcal{L}(c_{IJ}) \quad (1.3.81)$$

Since the corrections to α with respect to $\tilde{a}$ only start at order B^2 , it is possible to choose c_{IJ} a function of the λ s such that in this particular choice of scheme, we have:

$$\bar{\alpha}(s) = \tilde{a}(\lambda(\sqrt{s})) \quad (1.3.82)$$

On the other hand, the imaginary of α can only come from the non-local part of the dilaton effective action.

One can then show that:

$$-\text{Im}\alpha(s) = \frac{1}{16s^2} \sum_{\psi} (2\pi)^4 \delta(p_{\psi} - p_1 - p_2) B^I B^J \mathcal{M}_I(\psi) \mathcal{M}_J(\psi)^* \quad (1.3.83)$$

where we have defined:

$$B^I \mathcal{M}_I(\psi) \equiv B^I \langle \psi | \left[(\delta_K^I + \partial_I B^K) \mathcal{O}_K(0) + B^K \mathcal{O}_{IK}(p_1 - p_2) \right] | \Omega \rangle \quad (1.3.84)$$

and

$$\mathcal{O}_{IK}(p_1 - p_2) \equiv \int d^4 y e^{-i(p_1 - p_2)y/2} \mathbf{T} \{ \mathcal{O}_I(y) \mathcal{O}_J(-y) \} \quad (1.3.85)$$

We thus have proven the following formula:

$$-\text{Im}\alpha(s) = B^I B^J G_{IJ} \quad (1.3.86)$$

where

$$G_{IJ} = \frac{1}{16s^2} \sum_{\psi} (2\pi)^4 \delta(p_{\psi} - p_1 - p_2) \mathcal{M}_I(\psi) \mathcal{M}_J(\psi)^* \quad (1.3.87)$$

We can now plug this result back into our original dispersion relation which yields:

$$\bar{\alpha}(s_2) - \bar{\alpha}(s_1) = \frac{2}{\pi} \int_{s_1}^{s_2} \frac{ds}{s} B^I B^J G_{IJ} \quad (1.3.88)$$

To connect G_{IJ} and χ_{IJ}^g , let us start by integrating the gradient flow equation (1.3.38) which yields:

$$\tilde{a}(\lambda(\mu_2)) - \tilde{a}(\lambda(\mu_1)) = \frac{1}{8} \int_{\mu_1}^{\mu_2} \chi_{IJ}^g(\lambda(\mu)) B^I(\lambda(\mu)) B^J(\lambda(\mu)) \frac{d\mu}{\mu} \quad (1.3.89)$$

Now doing the change of variables $\mu = \sqrt{s}$ then gives:

$$\tilde{a}(\lambda(\sqrt{s_2})) - \tilde{a}(\lambda(\sqrt{s_1})) = \frac{1}{8} \int_{s_1}^{s_2} \chi_{IJ}^g(\lambda(\sqrt{s})) B^I(\lambda(\sqrt{s})) B^J(\lambda(\sqrt{s})) \frac{ds}{2s} \quad (1.3.90)$$

Identifying the right hand side of both equations then implies:

$$\chi_{IJ}^g = \frac{32}{\pi} G_{IJ} + \Delta_{IJ} \quad (1.3.91)$$

where Δ_{IJ} is a matrix such that

$$\Delta_{IJ} B^I B^J = 0 \quad (1.3.92)$$

We have therefore proven that χ_{IJ}^g is positive definite in perturbation theory (up to the matrix Δ).

The first consequence is the perturbative strong a-theorem. Indeed, the gradient flow formula (1.3.38) takes the form:

$$\mu \frac{d}{d\mu} \tilde{a} \equiv B^I \partial_I \tilde{a} = \frac{1}{8} \chi_{IJ}^g B^I B^J \quad (1.3.93)$$

which, due to the positive definiteness of χ_{IJ}^g , directly implies that $\tilde{a}$ is the monotonously increasing function we were looking for.

We have already mentioned that

$$\tilde{a}(\lambda(\mu_2)) - \tilde{a}(\lambda(\mu_1)) = \frac{1}{8} \int_{\mu_1}^{\mu_2} \chi_{IJ}^g(\lambda(\mu)) B^I(\lambda(\mu)) B^J(\lambda(\mu)) \frac{d\mu}{\mu} \quad (1.3.94)$$

has to be finite as long as the flow stays in the perturbative neighborhood of the fixed point. This will yield a straightforward bound on the RG flow asymptotics. Specifically, if the flow stays in the mentioned neighborhood asymptotically as $\ln \mu \rightarrow \pm\infty$, the *positive* integrand has to vanish, i.e.

$$\lim_{\ln \mu \rightarrow \pm\infty} \chi_{IJ}^g(\lambda(\mu)) B^I(\lambda(\mu)) B^J(\lambda(\mu)) = 0 \quad (1.3.95)$$

There are two conditions under which this can happen:

- The matrix $\chi_{IJ}^g(\lambda(\mu))$ tends to a matrix with vanishing eigenvalues. This possibility can be excluded because of the same reason as in the two dimensional case, namely because it would violate our perturbativity assumption.
- The beta function B^I vanish asymptotically.

We have therefore proven that under the assumption that the flow stays perturbative, its asymptotics are characterized by a vanishing beta function, i.e. the asymptotics of the flow have to be a CFT.

Chapter 2

Possible IR problems in the proof and ways out

2.1 Statement of the IR problems

In this section, we will discuss the possible problems in the various proofs of the previous chapter. It essentially boils down to two points:

1. Is the DSA well defined?
2. Does it make sense to talk about a UV theory in the forward limit which entails low momenta?

We start by discussing the first point. Whenever one studies “scattering” amplitudes or maybe more generally Green’s functions, there is always question of whether they are well defined, i.e. whether they are finite and independent of the cut-off. Indeed, they nearly always have UV divergences and sometimes they also have IR divergences. We have ways to deal with these problems (e.g. renormalization and inclusive cross-sections) but they entail a big amount of work. It turns out that for the DSA, the UV divergences are easy to handle. Indeed, we have argued in the previous chapter that if one imposes the on-shell condition for the canonical dilaton

$$\Box\phi = 0 \tag{2.1.1}$$

the DSA is automatically RG independent and therefore UV finite. We emphasize again that this is one of the crucial steps in the proof of the a-theorem in *four* dimensions. Now what about the IR limit? Here we could encounter several problems. Indeed, the on-shell condition can roughly be understood as taking the dilaton to be massless. Indeed, it forces us to take the external momenta of the DSA to be on the light-cone. This limit could in principle very well generate IR divergences. A possible second source of IR divergences is the forward limit, which we had to impose in our proof of the a-theorem to have a positivity condition. Again, this limit is an absolutely crucial step in the proof. The forward limit is also an IR limit since it entails taking a certain combination of momenta, namely t , to zero. As such, it could very well generate its own IR divergence. This could for example happen if the DSA were to follow a power law in s/t .

Let us now discuss the second point. As we have just mentioned, the forward limit is an intrinsically IR limit. This then begs the question of whether not it makes sense to take this limit when we are considering the UV theory. The arguments of the previous chapter

were based on having a clear-cut distinction between the UV and the IR fixed point, so that each were characterized by their own theory. Indeed, the main idea in the proof was that in the limit $s/m^2 \rightarrow \infty$ where m is the scale which controls the RG flow, the amplitude can be approximated by the UV theory. The problem is that a priori, the UV limit would entail $s \gg m^2$ as well as $t \gg m^2$. So concretely, the limit we have considered in the proof is a sort of mixed limit in which $s \gg m^2$ but $t \ll m^2$ and it is therefore a valid question whether in this limit we can actually approximate the DSA using only the UV theory.

In the two subsequent sections, we report some first attempts towards finding solutions. It needs to be emphasized that there are no clear results, maybe some will also turn out to be partially or totally wrong. The main purpose is to get the discussion of the previously mentioned problems rolling and to report on what has been tried so far.

2.2 Solution avenues for the forward limit

We first turn to the problems related to taking the forward limit. As we have already mentioned, at least in the UV, we are considering a mixed limit in which s goes to infinity but t goes to zero. It turns out that this limit is well-known as the Regge limit, which corresponds to s going to infinity while holding t fixed. This limit appears in what is called today Regge theory, which until Yang-Mills theory was developed, was an early attempt for a theory of the strong interaction. While its theoretical construction more or less failed, it remained on sure footing from an experimental point of view. Indeed, “Donnachie and Landshoff (1992) conclude their analysis of total cross sections based on Regge-type fits by stating that “Regge theory remains one of the great truths of particle physics”¹. Since the advent of Yang-Mills theory as the theory of the strong interaction, a great deal of effort by a great deal of people has been invested into showing that Yang-Mills theory indeed reproduces at least partially the results from Regge theory. The reason this is of interest to us is because part of the techniques involved in this is to consider only the leading logarithm, which generically is a $\ln s/t$, and then perform a resummation.

Coming back to our sheep, the original idea was now to try and compute the DSA in pure Yang-Mills (i.e. with only gluons) so as to be able to check explicitly whether the forward limit was well-defined or not. Let us recall that the DSA is equal to a linear combination of 2-,3- and 4-point functions of T the trace of the EM tensor and that T roughly obeys the following operator equation:

$$T = \beta^I \mathcal{O}_I \tag{2.2.1}$$

where $\mathcal{O}_I$ are the marginal operators of the theory. In the end, this means that computing the DSA comes down to computing 2-,3- and 4-point functions of the marginal operators of the theory (F^2 in pure Yang-Mills), which means trying to adapt the previously mentioned techniques to the case at hand. Before going over to the much more complicated case of Yang-Mills, we wanted to try to do the same thing in ϕ^3 theory, for which of course the operator ϕ^3 is not marginal. Already at his level, it turned out that adapting the techniques was non-trivial and the result we present in the following section are only related to ϕ^3 theory.

We will start by discussing very generally Regge theory. We then move on to present the standard techniques involved in extracting the Regge behavior from a simple theory,

¹Taken from chapter 5 of [17].

namely ϕ^3 . We then finally present some of the attempts to adapt the techniques towards computing the DSA.

2.2.1 Regge theory

Let us start by recalling some results from basic kinematics of a two-body process. In the center-of-mass (CM) frame, the process is described by two variables: $|\mathbf{p}|$, the CM momentum and the scattering angle in the s-channel θ_s (which is the angle between the momenta of the first and third particles). In what follows, we are going to be interested in the case where all the external particles have the same mass m . The relation between the CM variables and the Mandelstam variables s and t , then take a relatively simple form:

$$|\mathbf{p}|^2 = \frac{1}{4} (s - 4m^2) \quad (2.2.2)$$

$$\cos \theta_s = 1 + \frac{2t}{s - 4m^2} \quad (2.2.3)$$

The derivation of these formulas can be found for example in [17]. It is worth noting at this point that these formulas show us the physical domains of s and t . Indeed, we have that:

$$|\mathbf{p}|^2 \geq 0 \quad \Rightarrow \quad s - 4m^2 \geq 0 \quad (2.2.4)$$

$$-1 \leq \cos \theta_s \leq 1 \quad \Rightarrow \quad 0 \leq -t \leq s - 4m^2 \quad \Rightarrow \quad |t| \leq s - 4m^2 \quad (2.2.5)$$

Let us point one last feature of this result. In this work, we are in general going to be interested in the high-energy limit $s \gg m^2$, in which case the previous formulas simplify even further to:

$$|\mathbf{p}|^2 = \frac{1}{4} s \quad (2.2.6)$$

$$\cos \theta_s = 1 + \frac{2t}{s} \quad (2.2.7)$$

The next step is to do a partial-wave expansion of the scattering amplitude. We write the scattering amplitude in terms of the CM variables as $\tilde{A}$:

$$\tilde{A}(|\mathbf{p}|, \cos \theta_s) = \sum_{l=0} (2l+1) \tilde{A}_l(|\mathbf{p}|) P_l(\cos \theta_s) \quad (2.2.8)$$

This is the standard partial-wave expansion where P_l are the Legendre polynomials. The partial-wave amplitudes can be extracted using

$$\tilde{A}_l(s) = \frac{1}{2} \int_{-1}^1 d \cos \theta_s P_l(\cos \theta_s) \tilde{A}(|\mathbf{p}|, \cos \theta_s) \quad (2.2.9)$$

We then introduce the standard notation for scattering amplitudes in term of s and t :

$$A(s, t) \equiv \tilde{A} \left(\frac{1}{4} (s - 4m^2), 1 + \frac{2t}{s - 4m^2} \right) \quad \& \quad A_l(s) \equiv \tilde{A}_l \left(\frac{1}{4} (s - 4m^2) \right) \quad (2.2.10)$$

We now do a series of assumptions[17]:

1. We can continue the partial-wave amplitude $A_l(s)$ to complex values of l such that the resulting function $A(l, s)$ reduces to $A_l(s)$ for $l \in \mathbb{N}$

2. $A(l, s)$ only has isolated singularities in the complex l plane
3. $A(l, s)$ is a holomorphic function for $\text{Re}(l) \geq L$
4. If $\text{Re}(l) > 0$, we have $\lim_{|l| \rightarrow \infty} A(l, s) = 0$

For all of the details in what follows, refer to [17]. We will only report the results. Under these assumptions, it is then possible to prove that the amplitude can be rewritten in the Watson-Sommerfeld representation:

$$A(s, z) = - \sum_i \pi(2\alpha_i(s) + 1) \beta_i(s) \frac{P_{\alpha_i}(-z)}{\sin \pi \alpha_i} - \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} (2l+1) A(l, s) \frac{P_l(-z)}{\sin \pi l} dl \quad (2.2.11)$$

where $\alpha_i(s)$ is the location in the complex l plane of the i -th pole of $A(l, s)$, also called a Regge pole and $\beta_i(s)$ is the corresponding residue and $-\frac{1}{2} \leq \text{Re} \alpha_i < 0$. Using crossing symmetry and the asymptotic behavior of the Legendre polynomials, one can then finally prove that:

$$A(s, t) \xrightarrow{s \rightarrow \infty} -\beta(t) \frac{s^{\alpha(t)}}{\sin \pi \alpha(t)} \quad (2.2.12)$$

where $\alpha(s)$ is the right-most pole or equivalently the one for which $\text{Re}(\alpha_i)$ is the largest. Not much can be said about the corresponding residue $\beta(s)$. It turns out that in a relativistic setting this result has to be modified to:

$$A(s, t) \xrightarrow{s \rightarrow \infty} -\beta(t) \frac{1 + \xi e^{-i\pi \alpha(t)}}{\sin \pi \alpha(t)} s^{\alpha(t)} \quad (2.2.13)$$

where ξ is a new quantum number called the signature, which takes values $\xi = \pm 1$, which is necessary to make the analytic continuation possible.

Very generally, this can be interpreted as the amplitude for the exchange of “particle” of angular momentum $\alpha(t)$, which is called a Reggeon. The most general case one could then write down is the following. Imagine particles a and b scatter in the t -channel to particles c and d . Let $\gamma_{ac}(t)$ be the coupling of the Reggeon between particles a and c and similarly for $\gamma_{bd}(t)$. The amplitude would then go as:

$$A(s, t) \xrightarrow{s \rightarrow \infty} \frac{1 + \xi e^{-i\pi \alpha(t)}}{\sin \pi \alpha(t)} \frac{\gamma_{ac}(t) \gamma_{bd}(t)}{\Gamma(\alpha(t))} s^{\alpha(t)} \quad (2.2.14)$$

2.2.2 Leading logarithm contributions in a ϕ^3 theory

In this section, we are going to illustrate the techniques involved in extracting the leading logarithmic behavior of a simple theory. This section follows very closely chapter 2 of [18].

The theory we are going to consider is massive ϕ^3 , whose Lagrangian is given by

$$\mathcal{L} = (\partial\phi)^2 - m^2\phi^2 - \frac{g}{3!}\phi^3 \quad (2.2.15)$$

where g is the dimensionful coupling constant.

2.2.2.1 Leading order computation

It is clear that for the simple massive ϕ^3 we are considering, the leading order diagrams are the ones with the exchange of one scalar (see figure 2.1). We are going to neglect these diagrams, not because they are subleading, but because we are interested in the

logarithmic contributions and because ultimately this model is supposed to reproduce elastic quark scattering. For elastic quark scattering, we do not consider any change of color between the initial and the final quarks, imposing the exchange of a color singlet and therefore at least two gluons. This can be implemented in a simplified ϕ^3 scalar model as in [18] where one gauges the scalar field with an $SU(N) \times SU(N)$ gauge group. We do not consider this slight complication, because it does not add anything to the methods we are interested in, which are extracting the leading logarithmic behavior.

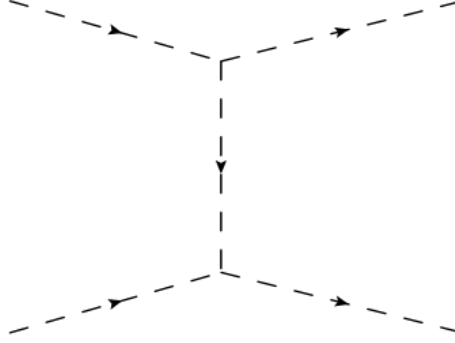


Figure 2.1: Leading order contribution to elastic scattering: t -channel.

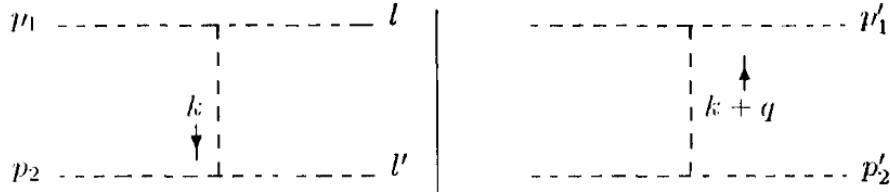


Figure 2.2: 0-rung diagram. (This image is taken from [18] and slightly modified.)

We therefore start with what we will call in the sense of the previous paragraph the “leading order” contribution whose Feynman diagram is shown in figure 2.2. To do so, we use the Cutkosky rules, which tell us that:

$$\text{Im}A = \frac{1}{2} \int d\Pi^{(2)} A_0(k) A_0^\dagger(k - q) \quad (2.2.16)$$

where A_0 is a tree-level amplitude and where

$$d\Pi^{(n)} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum k_i) d\Omega_1 \cdots d\Omega_n \quad (2.2.17)$$

is the Lorentz invariant phase-space (LIPS). We have also defined the $d\Omega_i$ which are the Lorentz invariant measures, i.e.

$$d\Omega_i = \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2E_i} = \frac{d^4 k_i}{(2\pi)^4} (2\pi) \delta^{(+)}(k_i^2 - m^2) \quad (2.2.18)$$

where the $\delta^{(+)}$ stands for the standard delta function multiplied by a theta function of the zero component, to ensure positivity of the energy.

Figure 2.2 yields:

$$A_0(k) = -g^2 \frac{1}{k^2 - m^2} \quad (2.2.19)$$

Let us now absorb some of the delta function and obtain:

$$\int d\Pi^{(2)} = \frac{1}{(2\pi)^2} \int d^4k \delta((p_1 - k)^2 - m^2) \delta((p_2 + k)^2 - m^2) \quad (2.2.20)$$

We now introduce the Sudakov parametrization of the momentum k :

$$k^\mu = \rho p_1^\mu + \lambda p_2^\mu + k_\perp^\mu \quad (2.2.21)$$

where $k_\perp^\mu$ is the momentum which is transverse to p_1 and p_2 , which actually is a two-dimensional vector, which we will represent by $\mathbf{k}$. Under this change of variables, the phase-space integral takes the following form:

$$\int d\Pi^{(2)} = \frac{s}{8\pi^2} \int d\rho d\lambda d^2\mathbf{k} \delta(-s(1-\rho)\lambda - \mathbf{k}^2 - m^2) \delta(s(1+\lambda\rho - \mathbf{k}^2 - m^2)) \quad (2.2.22)$$

The solution to the deltas are:

$$\begin{aligned} \rho_\pm &= \frac{s \pm \sqrt{(s)^2 - 4s(\mathbf{k}^2 - m^2)}}{2s} \\ \lambda_\pm &= \frac{-s \mp \sqrt{(s)^2 - 4s(\mathbf{k}^2 - m^2)}}{2s} \end{aligned}$$

Let us now compute k^2 and $(k - q)^2$. We have:

$$k^2 = \rho\lambda s - \mathbf{k}^2 \quad (2.2.23)$$

Now, using the solutions, we have:

$$\begin{aligned} \rho_- \lambda_- s &= \frac{1}{2} \left(2\mathbf{k}^2 - 2m^2 - s + \sqrt{(+s)^2 - 4s(\mathbf{k}^2 - m^2)} \right) \xrightarrow{s \rightarrow \infty} 0 \\ \rho_+ \lambda_+ s &= \frac{1}{2} \left(2\mathbf{k}^2 - 2m^2 - s - \sqrt{(+s)^2 - 4s(\mathbf{k}^2 - m^2)} \right) \xrightarrow{s \rightarrow \infty} -\infty \end{aligned}$$

This means that the $+$ -solutions are suppressed at high energies. Now at high energies, we have:

$$|\rho_- \lambda_- s| \approx \left| \frac{\mathbf{k}^2(2m^2 - \mathbf{k}^2) + m^4}{s} \right| \ll \mathbf{k}^2 + m^2 \quad (2.2.24)$$

Here we have used the fact that $\rho_- \lambda_- s$ appears only in the denominators of A_0 next to $\mathbf{k}^2 + m^2$. We therefore have that

$$k^2 \approx -\mathbf{k}^2 \quad (2.2.25)$$

Now, let us treat the transfer momentum:

$$\begin{aligned} 0 &= (p_1 - q)^2 = -(1 - \alpha)\beta s - \mathbf{q}^2 \\ 0 &= (p_2 + q)^2 = \alpha(1 + \beta)s - \mathbf{q}^2 \end{aligned}$$

The solutions are:

$$\begin{aligned} \alpha &= -\beta \\ \alpha_\pm &= \frac{1 \pm \sqrt{1 - 4\frac{\mathbf{q}^2}{s}}}{2} \end{aligned}$$

In the limit of $\mathbf{q}^2 \ll s$, we then have:

$$\begin{aligned}\alpha_+ &\approx 1 - \frac{\mathbf{q}^2}{s} \\ \alpha_- &\approx \frac{\mathbf{q}^2}{s}\end{aligned}$$

We then find:

$$\begin{aligned}\alpha_- \beta_- s &\approx -\frac{\mathbf{q}^4}{s} \xrightarrow{s \rightarrow \infty} 0 \\ \alpha_+ \beta_+ s &\approx 2\mathbf{q}^2 - s \xrightarrow{s \rightarrow \infty} -\infty\end{aligned}$$

So again because of suppression, we choose the $--$ -solution and therefore obtain:

$$q^2 = -\mathbf{q}^2 - \frac{\mathbf{q}^4}{s} \quad (2.2.26)$$

Note that this implies that:

$$t \approx -\mathbf{q}^2 \quad (2.2.27)$$

Furthermore, we also have:

$$\begin{aligned}2\mathbf{k} \cdot \mathbf{q} &= \alpha_- (\lambda_- - \rho_-) s - 2\mathbf{k} \cdot \mathbf{q} \\ &\approx \frac{\mathbf{q}^2}{s} \left(-1 + \frac{\sqrt{(+s)^2 - 4s(\mathbf{k}^2 - m^2)}}{s} \right) s - 2\mathbf{k} \cdot \mathbf{q} \\ &\approx \frac{\mathbf{q}^2}{s} (2m^2 - 2\mathbf{k}^2) - 2\mathbf{k} \cdot \mathbf{q} \quad \approx -2\mathbf{k} \cdot \mathbf{q}\end{aligned}$$

So that finally, we have:

$$\begin{aligned}(k - q)^2 &= k^2 + q^2 - 2\mathbf{k} \cdot \mathbf{q} \\ &= -(\mathbf{k} - \mathbf{q})^2\end{aligned}$$

To summarize, we have proven that:

$$k^2 \approx -\mathbf{k}^2 \quad (2.2.28)$$

$$(k - q)^2 \approx -(\mathbf{k} - \mathbf{q})^2 \quad (2.2.29)$$

We therefore end up with:

$$\text{Im}A = \frac{g^4}{16\pi^2 s} \int d^2\mathbf{k} \frac{1}{(\mathbf{k}^2 + m^2)} \frac{1}{((\mathbf{k} - \mathbf{q})^2 + m^2)} \quad (2.2.30)$$

One of the most important points to notice about the form of the result is that the denominator structure is exactly that of the product of two Lorentzians, so that this integral is clearly dominated by the region where $\mathbf{k}$ is of the order of the large of m and $\sqrt{|t|}$.

2.2.2.2 Next-to-leading order computation

We now perform the next-to-leading order computation for which we consider the diagram shown in figure 2.3. Here again, we have:

$$\text{Im}A = \frac{1}{2} \int d\Pi^{(3)} A_1(k) A_1(k-q) \quad (2.2.31)$$

where figure 2.3 gives us

$$A_1 = \frac{g^3}{(k_1^2 - m^2)(k_2^2 - m^2)} \quad (2.2.32)$$

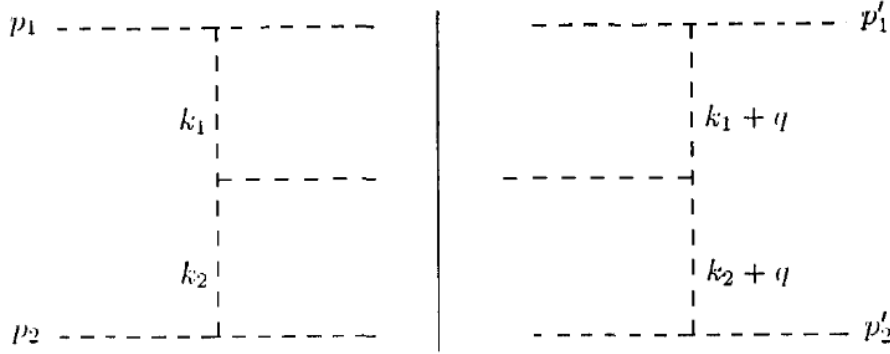


Figure 2.3: 1-rung diagram. (This image is taken from [18] and slightly modified.)

The change of variables to the Sudakov parametrization then yields:

$$\begin{aligned} \int d\Pi^{(3)} &= \frac{s^2}{128\pi^5} \int d\rho_1 d\lambda_1 d^2\mathbf{k}_1 d\rho_2 d\lambda_2 d^2\mathbf{k}_2 \\ &\times \delta(-s(1-\rho_1)\lambda_1 - (\mathbf{k}_1^2 + m^2)) \delta(s(1+\lambda_2)\rho_2 - (\mathbf{k}_2^2 + m^2)) \\ &\times \delta(s(\rho_1 - \rho_2)(\lambda_1 - \lambda_2) - ((\mathbf{k}_1 - \mathbf{k}_2)^2 + m^2)) \end{aligned} \quad (2.2.33)$$

In the limit we are considering, we expect again for the same reasons as before that the denominator structure takes the form of Lorentzians which then implies that all the transverse momenta will essentially be of the order of the larger of m and $\sqrt{|t|}$. The logarithms of s are then going to be scaled by these transverse momenta, but to leading logarithmic order we can make the approximation:

$$\mathbf{k}_1^2 = \mathbf{k}_2^2 = (\mathbf{k}_1 - \mathbf{k}_2)^2 = \mathbf{k}^2 \quad (2.2.34)$$

where $\mathbf{k}$ is a stand-in vector whose magnitude is of the order of the larger of m and $\sqrt{|t|}$.

There is first set of simple constraints which arises from the delta functions:

$$1 > \rho_1 > \rho_2 > 0 \quad \& \quad 1 > |\lambda_2| > |\lambda_1| > 0 \quad (2.2.35)$$

where the λ s are negative. This is a direct of consequence of the fact that the $\mathbf{k}^2 + m^2$ in the delta functions are positive and therefore constrain the ρ s and the λ s. Now logarithms appear when there is a separation of scales, so that the above inequalities are sharpened to the strong ordering approximation:

$$1 \gg \rho_1 \gg \rho_2 \quad \& \quad 1 \gg |\lambda_2| \gg |\lambda_1| \quad (2.2.36)$$

The fact that this domain yields the leading logarithmic order is essentially an assumption, whose validity can be verified at the end of the computation. The phase-space integral then reduces to

$$\begin{aligned} \int d\Pi^{(3)} &= \frac{s^2}{128\pi^5} \int d\rho_1 d\lambda_1 d^2\mathbf{k}_1 d\rho_2 d\lambda_2 d^2\mathbf{k}_2 \\ &\quad \times \delta(-s\lambda_1 - (\mathbf{k}^2 + m^2)) \delta(s\rho_2 - (\mathbf{k}^2 + m^2)) \\ &\quad \times \delta(-s\rho_1\lambda_2 - (\mathbf{k}^2 + m^2)) \end{aligned} \quad (2.2.37)$$

We now do the integrations over λ_1 and λ_2 yielding:

$$\int d\Pi^{(3)} = \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} d\rho_2 d^2\mathbf{k}_1 d^2\mathbf{k}_2 \delta(s\rho_2 - \mathbf{k}^2) \quad (2.2.38)$$

The third delta imposes that

$$s\rho_1\lambda_2 \approx \mathbf{k}^2 + m^2 \quad (2.2.39)$$

which implies that in the strong ordering regime we have:

$$s\rho_1\lambda_1 \ll \mathbf{k}_1^2 \quad \& \quad s\rho_2\lambda_2 \ll \mathbf{k}_2^2 \quad (2.2.40)$$

so that finally A_1 reduces to

$$A_1(k) = \frac{g^3}{(\mathbf{k}_1^2 + m^2)(\mathbf{k}_2^2 + m^2)} \quad (2.2.41)$$

So that finally we have:

$$\text{Im}A = \frac{g^4}{16\pi^2 s} \int d^2\mathbf{k}_1 f_1(s, \mathbf{k}_1, \mathbf{q}) \quad (2.2.42)$$

where we have defined

$$\begin{aligned} f_1(s, \mathbf{k}_1, \mathbf{q}) &= \frac{g^2 s}{2(2\pi)^3} \int_0^1 d\rho_2 \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \delta(s\rho_2 - \mathbf{k}^2) \int d^2\mathbf{k}_2 \\ &\quad \times \frac{1}{(\mathbf{k}_1^2 + m^2)(\mathbf{k}_2^2 + m^2)} \frac{1}{((\mathbf{k}_1 - \mathbf{q})^2 + m^2)((\mathbf{k}_2 - \mathbf{q})^2 + m^2)} \end{aligned} \quad (2.2.43)$$

Notice that we factored out s for dimensional reasons, i.e. f_1 is now dimensionless.

To simplify the computations and to have clear way to extract the leading logarithmic contribution, we now take the Mellin transform of f_1 defined as:

$$\mathcal{F}_1(\omega, \mathbf{k}_1, \mathbf{q}) = \int_1^\infty d\left(\frac{s}{\mathbf{k}^2}\right) \left(\frac{s}{\mathbf{k}^2}\right)^{-\omega-1} f_1(s, \mathbf{k}_1, \mathbf{q}) \quad (2.2.44)$$

Performing the integration over s by absorbing the delta function and making the change of variables

$$\tau_1 = \rho_1 \quad \& \quad \tau_1\tau_2 = \rho_2 \quad (2.2.45)$$

then yields:

$$\begin{aligned} \mathcal{F}_1(\omega, \mathbf{k}_1, \mathbf{q}) &= \frac{g^2 s}{2(2\pi)^3} \int_0^1 d\tau_1 \tau_1^{\omega-1} \int_0^1 d\tau_2 \tau_2^{\omega-1} \int d^2 \mathbf{k}_2 \\ &\quad \times \frac{1}{(\mathbf{k}_1^2 + m^2)(\mathbf{k}_2^2 + m^2)} \frac{1}{((\mathbf{k}_1 - \mathbf{q})^2 + m^2)((\mathbf{k}_2 - \mathbf{q})^2 + m^2)} \end{aligned} \quad (2.2.46)$$

$$\begin{aligned} \Rightarrow \quad \omega^2 \mathcal{F}_1(\omega, \mathbf{k}_1, \mathbf{q}) &= \frac{g^2 s}{2(2\pi)^3} \int d^2 \mathbf{k}_2 \\ &\quad \times \frac{1}{(\mathbf{k}_1^2 + m^2)(\mathbf{k}_2^2 + m^2)} \frac{1}{((\mathbf{k}_1 - \mathbf{q})^2 + m^2)((\mathbf{k}_2 - \mathbf{q})^2 + m^2)} \end{aligned} \quad (2.2.47)$$

The point of doing the Mellin transform is that poles in the Mellin transform correspond to logarithms after inverting the Mellin transform. More precisely, we have the following Mellin pair:

$$f(s) = (\ln(s))^r s^\alpha \quad \& \quad \mathcal{F}(\omega) = \mathbf{k}^2 \frac{\Gamma(r+1)}{(\omega - \alpha)^{r+1}} \quad (2.2.48)$$

One can now also argue why other diagrams do not contribute to the this leading logarithmic computation. Any diagrams with vertex or self-energy insertions (see figure 2.4) cannot contribute to this order simply because they cannot depend on s , therefore making it impossible for them to contribute the sought after $\ln s$. Another diagram which does not contribute is the vertex correction diagram but with three cut lines (see figure 2.5). This is because the momenta are still ordered as they were before, so that one of the denominators is of the order of $|\lambda_2|s$ which is much larger than $\mathbf{k}^2$, because from the previous conditions we know that $|\lambda_2| \ll \frac{\mathbf{k}^2}{s}$. This large denominator then suppresses this diagram. Finally, we have the three gluon diagrams which can also be shown to be subleading (see figure 2.6).

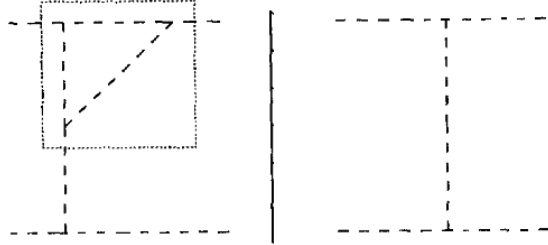


Figure 2.4: Vertex or self-energy insertion. (This image is taken from [18] and slightly modified.)

2.2.2.3 The n -rung ladder diagram

The generalization to the n -rung ladder diagram is now a simple generalization. We again have:

$$\text{Im}A = \frac{1}{2} \int d\Pi^{(3)} A_n(k) A_n(k - q) \quad (2.2.49)$$

with the diagram given in figure 2.7 which yields

$$A_n(k) = (-g)^{n+2} \Pi_{i=1}^{n+1} \frac{1}{k_i^2 - m^2} \quad (2.2.50)$$

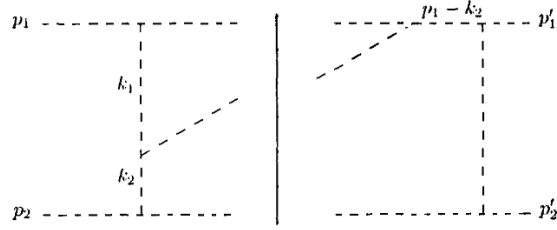


Figure 2.5: Cut vertex or self-energy insertion. (This image is taken from [18] and slightly modified.)



Figure 2.6: Exchange of 3 t -channel particles. (This image is taken from [18] and slightly modified.)

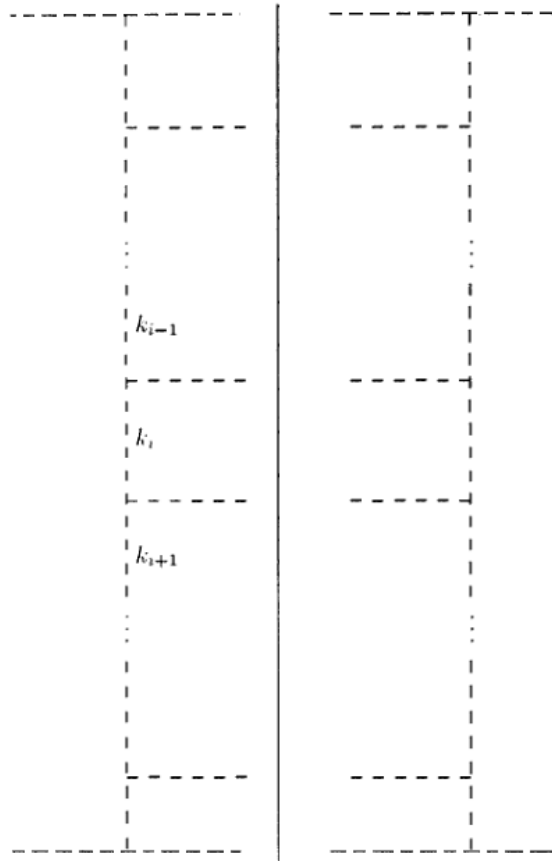


Figure 2.7: n -rung ladder diagram. (This image is taken from [18] and slightly modified.)

The phase-space integral now takes the form:

$$\int d\Pi^{(n+2)} = \frac{s^{n+1}}{2^{4n+3}\pi^{3n+2}} \int \prod_{i=1}^{n+1} d\rho_i d\lambda_i d^2\mathbf{k}_i \prod_{j=1}^n \delta(s(\rho_j - \rho_{j+1})(\lambda_j - \lambda_{j+1}) - ((\mathbf{k}_j - \mathbf{k}_{j+1})^2 + m^2)) \\ \delta(-s(1 - \rho_1)\lambda_1 - (\mathbf{k}_1^2 + m^2)) \delta(s(1 + \lambda_{n+1})\rho_{n+1} - (\mathbf{k}_{n+1}^2 + m^2)) \quad (2.2.51)$$

where we have already performed the change of variables to the Sudakov parametrization.

We also still have that the phase-space integral will be dominated by region where the transverse momenta are all of the same order which is the larger of m^2 and $|t|$ and that strong ordering implies:

$$A_n(k) = -g^{n+2} \prod_{i=1}^{n+1} \frac{1}{\mathbf{k}_i^2 + m^2} \quad (2.2.52)$$

The same steps as before then yield:

$$\text{Im}A = \frac{g^4}{16\pi^2 s} \int d^2\mathbf{k}_1 f_n(s, \mathbf{k}_1, \mathbf{q}) \quad (2.2.53)$$

where we have defined:

$$f_n(s, \mathbf{k}_1, \mathbf{q}) = \frac{g^{2n}}{2^{4n}\pi^{3n}} \prod_{j=2}^{n+1} \int d^2\mathbf{k}_j \prod_{i=1}^n \int_{\rho_{i+1}}^1 \frac{d\rho_i}{\rho_i} \int_0^1 d\rho_{n+1} \quad (2.2.54)$$

$$\times \prod_{m=1}^{n+1} \frac{1}{(\mathbf{k}_m^2 + m^2)((\mathbf{k}_m - \mathbf{q})^2 + m^2)} s \delta(s\rho_{n+1} - (\mathbf{k}^2 + m^2)) \quad (2.2.55)$$

As before, we now take the Mellin transform and make the change of variables

$$\tau_i = \frac{\rho_i}{\rho_{i-1}} \quad (2.2.56)$$

which unravels the nested integrals yielding:

$$\omega^{n+1} \mathcal{F}_n(\omega, \mathbf{k}_1, \mathbf{q}) = \frac{g^{2n}}{2^{4n}\pi^{3n}} \left(\int d^2\mathbf{k} \frac{1}{(\mathbf{k}^2 + m^2)((\mathbf{k} - \mathbf{q})^2 + m^2)} \right)^n \frac{1}{(\mathbf{k}_1^2 + m^2)((\mathbf{k}_1 - \mathbf{q})^2 + m^2)} \quad (2.2.57)$$

The crucial result here is that $\mathcal{F}_n$ has a pole of order $n+1$, whose inverse Mellin transform then yields $\frac{\ln^n s}{n!}$.

Let us mention that one can again argue that crossed ladder diagrams do not contribute due to denominator suppression. For barely more details, see [18].

Let us now perform the resummation:

$$\mathcal{F}(\omega, \mathbf{k}, \mathbf{q}) = \sum_{n=0}^{\infty} \mathcal{F}_n(\omega, \mathbf{k}, \mathbf{q}) = \frac{1}{(\mathbf{k}^2 + m^2)((\mathbf{k} - \mathbf{q})^2 + m^2)(\omega - 1 - \alpha_P(t))} \quad (2.2.58)$$

where

$$\alpha_P(t) = -1 + \frac{g^2}{2(2\pi)^3} \int d^2\mathbf{k} \frac{1}{(\mathbf{k}^2 + m^2)((\mathbf{k} - \mathbf{q})^2 + m^2)} \quad (2.2.59)$$

$$\begin{aligned} &= -1 + \frac{\pi g^2}{q^2 2(2\pi)^3} \int_0^1 dx \frac{1}{x(1-x) + \frac{m^2}{q^2}} \\ &\approx -1 + \frac{\pi g^2}{q^2 (2\pi)^3} \int_0^1 dx \left(\frac{q^2}{m^2} - \left(\frac{q^2}{m^2} \right)^2 x(1-x) + O\left(\left(\frac{q^2}{m^2} \right)^3 \right) \right) \\ &= -1 + \frac{g^2}{m^2 16\pi^2} \left(1 - \frac{q^2}{6m^2} \right) \\ &= -1 + \frac{g^2}{m^2 16\pi^2} \left(1 + \frac{t}{6m^2} \right) \end{aligned} \quad (2.2.60)$$

where in the third line we took $|t| \ll m^2$ and where we used results we present in the next section, namely (2.2.86).

For completeness, let us mention that one can now use a dispersion relation to obtain the complete amplitude which turns out to be:

$$A(s, t) = \frac{C}{\pi(1 + \alpha_P(t))} \left(\left(\frac{s}{t} \right)^{1 + \alpha_P(t)} + \left(\frac{u}{t} \right)^{1 + \alpha_P(t)} \right) \quad (2.2.61)$$

where the second contribution comes from an overall crossing where we simply exchange $u \leftrightarrow s$.

We now want to emphasize a point because this is in the end the only significant thing which will change when we go to the DSA. In the previous computation, we always approximated $k^2 - m^2 \approx -\mathbf{k}^2 - m^2$, meaning that we dropped the longitudinal part. The reason this is legitimate in this computation is because the longitudinal part is comprised of terms which go like m^2/s or t/s , which are subleading compared to $-\mathbf{k}^2 - m^2$. This is going to change in the next section.

2.2.3 Modifying the Pomeron method for the dilaton amplitude

In this section, we try to adapt the techniques illustrated in the previous section. The idea is to consider a simple massive ϕ^3 theory and compute the four-point function of ϕ^3 , namely $\langle \phi^3(x_1) \phi^3(x_2) \phi^3(x_3) \phi^3(x_4) \rangle$. It turns out that the computation seemingly looks very similar, but in the end is quite different. The point is that for the DSA one can look at exactly the same diagram than before except that the initial and final rungs have to be replaced by a loop. The idea we introduced to deal with this additional loop is to replace it again by a single particle with a given mass which in the end has to be integrated over at the end.

2.2.3.1 Leading order ϕ^3 four-point function

In this section, we will compute the leading order result for the DSA. The techniques used are very similar to those used in the previous section. The dominant leading order graph we will consider is shown in figure 2.8. There are of course other types of diagrams, some of which are shown in figures 2.9 and 2.10, which we believe to be subleading, but which have not been explored in more detail.

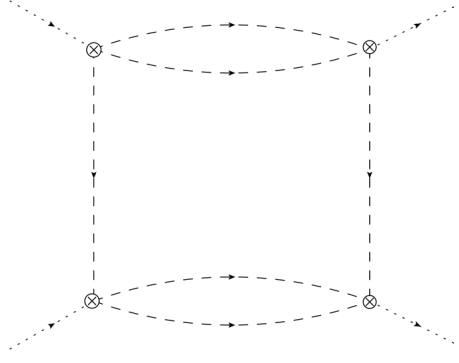


Figure 2.8: Dominant leading order Feynman diagram for the DSA.

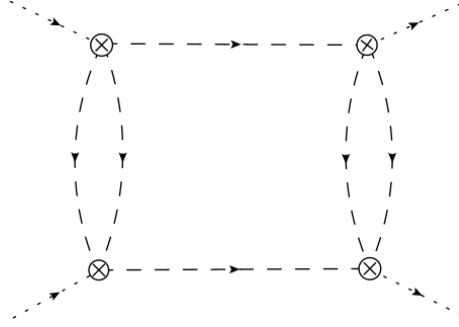


Figure 2.9: Other leading order Feynman diagram for the DSA.

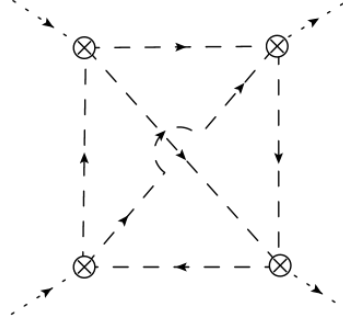


Figure 2.10: Other leading order Feynman diagram for the DSA.

2.2.3.2 Leading order computation for $m = 0$ and $t = 0$

Let us start with the truly most simple case with massless particles and $t = 0$. The cutting rules then dictate:

$$2\text{Im}(\mathcal{A}) = \int d\Pi^{(4)} |\tilde{\mathcal{A}}|^2 \quad (2.2.62)$$

where

$$d\Pi^{(4)} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum k_i) d\Omega_1 \cdots d\Omega_4 \quad (2.2.63)$$

is the Lorentz invariant phase-space (LIPS). We have also defined the $d\Omega_i$ which are the Lorentz invariant measures, i.e.

$$d\Omega_i = \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2E_i} = \frac{d^4 k_i}{(2\pi)^4} (2\pi) \delta^{(+)}(k_i^2) \quad (2.2.64)$$

where the $\delta^{(+)}$ stands for the standard delta function multiplied by a theta function of the zero component, to ensure positivity of the energy.

The idea is now to replace each two outgoing particles by one massive particle. This only works if the rest of the integral depends only on some specific combination of the momenta k_i , e.g. on $k_1 + k_2$ and $k_3 + k_4$. To do so, we multiply by the identity in the following form:

$$1 = \int \frac{dM_A^2}{2\pi} (2\pi) \delta^{(+)}((k_1 + k_2)^2 - M_A^2) \int \frac{dM_B^2}{2\pi} (2\pi) \delta^{(+)}((k_3 + k_4)^2 - M_B^2) \\ \cdot \int \frac{d^4 k_A}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k_A - k_1 - k_2) \int \frac{d^4 k_B}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k_B - k_3 - k_4)$$

By introducing this special factor of 1, we can then factorize out of $d\Pi^{(4)}$ the following two factors:

$$\int d\Omega_1 d\Omega_2 (2\pi)^4 \delta^{(4)}(k_A - k_1 - k_2) \quad \& \quad \int d\Omega_3 d\Omega_4 (2\pi)^4 \delta^{(4)}(k_B - k_3 - k_4) \quad (2.2.65)$$

These two factors are now simply the phase-space for a two body decay. Since $k_1^2 = k_2^2 = 0$, we then have:

$$\int d\Omega_1 d\Omega_2 (2\pi)^4 \delta^{(4)}(k_A - k_1 - k_2) = \frac{1}{8\pi} \quad \& \quad \int d\Omega_3 d\Omega_4 (2\pi)^4 \delta^{(4)}(k_B - k_3 - k_4) = \frac{1}{8\pi} \quad (2.2.66)$$

Notice that if the theory was massive, the above result would be slightly modified to

$$\int d\Omega_1 d\Omega_2 (2\pi)^4 \delta^{(4)}(k_A - k_1 - k_2) = \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{M_A^2}} \\ \& \quad \int d\Omega_3 d\Omega_4 (2\pi)^4 \delta^{(4)}(k_B - k_3 - k_4) = \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{M_B^2}} \quad (2.2.67)$$

The reason we can do these integrals without further ado is that the rest of integral, specifically $\tilde{\mathcal{A}}$ does not depend on k_i individually but only on k_A and k_B . We have now obtained:

$$2\text{Im}(\mathcal{A}) = \int \frac{dM_A^2}{2\pi} \int \frac{dM_B^2}{2\pi} \int \frac{d^4 k_A}{(2\pi)^4} (2\pi) \delta^{(+)}((k_A)^2 - M_A^2) \int \frac{d^4 k_B}{(2\pi)^4} (2\pi) \delta^{(+)}((k_B)^2 - M_B^2) \\ \cdot (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_A - k_B) \frac{1}{(8\pi)^2} |\tilde{\mathcal{A}}|^2$$

This finally reduces to:

$$2\text{Im}(\mathcal{A}) = \int \frac{dM_A^2}{2\pi} \int \frac{dM_B^2}{2\pi} \frac{1}{(8\pi)^2} \int d\Pi_{AB}^{(2)} \left| \frac{1}{(p_1 - k_A)^2} \right|^2 \quad (2.2.68)$$

Let us find an expression for $(p_1 - k_A)^2$:

$$\begin{aligned}
t_{AB} &\equiv (p_1 - k_A)^2 = M_A^2 - 2p_1 \cdot k_A \\
&= M_A^2 - 2(E_1 E_A - |\mathbf{p}'| |\mathbf{p}'| \cos \theta) \\
&= M_A^2 - 2 \left(\frac{\sqrt{s}}{2} \frac{1}{2\sqrt{s}} (s + M_A^2 - M_B^2) - \frac{\sqrt{s}}{2} \frac{\sqrt{\lambda(s, M_A^2, M_B^2)}}{2\sqrt{s}} \cos \theta \right) \\
&= M_A^2 - \frac{1}{2} \left(s + M_A^2 - M_B^2 - \sqrt{\lambda(s, M_A^2, M_B^2)} \cos \theta \right) \\
&= \frac{\sqrt{\lambda}}{2} \left(\cos \theta + \frac{1}{\sqrt{\lambda}} (M_A^2 + M_B^2 - s) \right)
\end{aligned}$$

where we have used

$$|\mathbf{p}'|^2 = \frac{1}{4s} \lambda(s, m_A^2, m_B^2) \quad \& \quad |\mathbf{p}|^2 = \frac{1}{4s} \lambda(s, 0, 0) = \frac{s}{4} \quad (2.2.69)$$

and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ is the triangle function.

We have:

$$\begin{aligned}
\int d\Pi_{AB}^{(2)} \left| \frac{1}{(p_1 - k_A)^2} \right|^2 &= \int \frac{d\Omega}{16\pi^2} \frac{|\mathbf{p}'|}{\sqrt{s}} \frac{1}{(t_{AB})^2} \\
&= \frac{1}{8\pi} \frac{|\mathbf{p}'|}{\sqrt{s}} \int_{-1}^1 dx \frac{1}{\frac{\lambda}{4} \left(x + \frac{1}{\sqrt{\lambda}} (M_A^2 + M_B^2 - s) \right)^2} \\
&= \frac{1}{8\pi} \frac{|\mathbf{p}'|}{\sqrt{s}} \frac{2}{\frac{\lambda}{4} \left(\frac{1}{\lambda} (M_A^2 + M_B^2 - s)^2 - 1 \right)} \\
&= \frac{1}{\pi} \frac{|\mathbf{p}'|}{\sqrt{s}} \frac{1}{(M_A^2 + M_B^2 - s)^2 - \lambda} \\
&= \frac{1}{\pi} \frac{|\mathbf{p}'|}{\sqrt{s}} \frac{1}{4M_A^2 M_B^2} \\
&= \frac{1}{4\pi M_A^2 M_B^2} \frac{|\mathbf{p}'|}{\sqrt{s}}
\end{aligned}$$

We are then finally left with:

$$\begin{aligned}
2\text{Im}(\mathcal{A}) &= \frac{1}{2^8 \pi^4 2^2 \pi} \int dM_A^2 \int dM_B^2 \frac{|\mathbf{p}'|}{M_A^2 M_B^2 \sqrt{s}} \\
&= \frac{1}{2^{10} \pi^5} \int dM_A^2 \int dM_B^2 \frac{\sqrt{\lambda(s, m_A^2, m_B^2)}}{2\sqrt{s} M_A^2 M_B^2 \sqrt{s}} \\
&= \frac{1}{2^{11} \pi^5 s} \int dM_A^2 \int dM_B^2 \frac{\sqrt{[s - (M_A + M_B)^2][s - (M_A - M_B)^2]}}{M_A^2 M_B^2} \\
&= \frac{1}{2^9 \pi^5 s} \int dM_A \int dM_B \frac{\sqrt{[s - (M_A + M_B)^2][s - (M_A - M_B)^2]}}{M_A M_B}
\end{aligned}$$

We have therefore obtained that at the leading order in the massless and $t = 0$ limit is divergent, but the form of the integral already let us foresee that the result for $t \neq 0$ will be

a logarithm of t . Moreover, this computation already points towards how the computation in the DSA case could work. Intuitively, from looking at the diagrams we are considering for the DSA, it seems that it should not deviate too much from the original computation. Indeed, “all” we did was double the initial and final rungs, which in some sense amounts to adding another rung but at coincident points with the initial rung (respectively the last rung). The original computation essentially shows that each rung yields a factor of $\ln s$. It is therefore a sensible hypothesis that the DSA result would be proportional to $\ln^{n+2} s$, since we “simply” added two rungs. How do we get these extra logarithm factors in our computation? Through the $1/M$ integrals. To sum up, already this very simple computation does give us an idea of what to expect, of what to strive for in the following mess of computations.

2.2.3.3 Leading order computation for $m = 0$ and $t \neq 0$

Let us now repeat the previous computation with $t \neq 0$ for which we simply need to replace $|\tilde{\mathcal{A}}|^2$ by

$$\tilde{\mathcal{A}}\tilde{\mathcal{A}}^\dagger = \frac{1}{(p_1 - k_A)^2} \frac{1}{(p_3 - k_A)^2} = \frac{1}{(p_1 - k_A)^2} \frac{1}{(p_1 - k_A - q)^2} \quad (2.2.70)$$

where $q = p_1 - p_3 = p_4 - p_2$ is the transfer momentum. Except for that minor change, everything goes similarly for a while and we end up with:

$$2\text{Im}(\mathcal{A}) = \int \frac{dM_A^2}{2\pi} \int \frac{dM_B^2}{2\pi} \frac{1}{(8\pi)^2} \left(\int d\Pi_{AB}^{(2)} \frac{1}{(p_1 - k_A)^2} \frac{1}{(p_1 - k_A - q)^2} \right) \quad (2.2.71)$$

The term between brackets is exactly what we would have obtained by cutting a standard box diagram where the horizontal particles are massive and the vertical ones are massless. Now, we rewrite $d\Pi_{AB}^{(2)}$ in term of loop momenta as

$$\int d\Pi_{AB}^{(2)} \frac{1}{(p_1 - k_A)^2} \frac{1}{(p_1 - k_A - q)^2} = \frac{1}{(2\pi)^2} \int d^4k \delta((p_1 - k)^2 - M_A^2) \delta((p_2 + k)^2 - M_B^2) \frac{1}{k^2(k - q)^2} \quad (2.2.72)$$

where we have made the change of variables $k_A = p_1 - k$. Introducing the Sudakov parametrization, we obtain:

$$\begin{aligned} & \int d\Pi_{AB}^{(2)} \frac{1}{(p_1 - k_A)^2} \frac{1}{(p_1 - k_A - q)^2} \\ &= \frac{s}{8\pi^2} \int d\rho d\lambda d^2\mathbf{k} \delta(-s(1 - \rho)\lambda - (\mathbf{k}^2 + M_A^2)) \delta(s(1 + \lambda)\rho - (\mathbf{k}^2 + M_B^2)) \frac{1}{k^2(k - q)^2} \end{aligned} \quad (2.2.73)$$

The computations in the following separated section are nearly identical to those in the previous section about the original computation. The main difference now is that we have two different masses and that we will keep their contribution, because contrary to the previous section, the masses of the first and last rung, which are now being integrated over, are not subleading anymore.

* * * * *

The solution to the deltas are:

$$\rho_{\pm} = \frac{M_A^2 - M_B^2 + s \pm \sqrt{(M_A^2 - M_B^2 + s)^2 - 4s(\mathbf{k}^2 - M_B^2)}}{2s}$$

$$\lambda_{\pm} = \frac{M_A^2 - M_B^2 - s \mp \sqrt{(M_A^2 - M_B^2 + s)^2 - 4s(\mathbf{k}^2 - M_B^2)}}{2s}$$

Let us now calculate k^2 and $(k - q)^2$. We have:

$$k^2 = \rho\lambda s - \mathbf{k}^2 \quad (2.2.74)$$

Now, using the solutions, we have:

$$\rho_- \lambda_- s = \frac{1}{2} \left(2\mathbf{k}^2 - M_A^2 - M_B^2 - s + \sqrt{(M_A^2 - M_B^2 + s)^2 - 4s(\mathbf{k}^2 - M_B^2)} \right) \xrightarrow{s \rightarrow \infty} 0$$

$$\rho_+ \lambda_+ s = \frac{1}{2} \left(2\mathbf{k}^2 - M_A^2 - M_B^2 - s - \sqrt{(M_A^2 - M_B^2 + s)^2 - 4s(\mathbf{k}^2 - M_B^2)} \right) \xrightarrow{s \rightarrow \infty} -\infty$$

This means that the $+$ -solutions are suppressed at high energies. Now at high energies, we have:

$$\rho_- \lambda_- s \approx \frac{\mathbf{k}^2(M_A^2 + M_B^2 - \mathbf{k}^2) + M_A^2 M_B^2}{s} \approx -\frac{M_A^2 M_B^2}{s} \quad (2.2.75)$$

where in the second approximation we used the fact that $\mathbf{k}^2 \approx t$. We therefore have that

$$k^2 \approx -\mathbf{k}^2 - \frac{M_A^2 M_B^2}{s} \quad (2.2.76)$$

Now, let us treat the transfer momentum:

$$0 = (p_1 - q)^2 = -(1 - \alpha)\beta s - \mathbf{q}^2$$

$$0 = (p_2 + q)^2 = \alpha(1 + \beta)s - \mathbf{q}^2$$

The solutions are:

$$\alpha = -\beta$$

$$\alpha_{\pm} = \frac{1 \pm \sqrt{1 - 4\frac{\mathbf{q}^2}{s}}}{2}$$

In the limit of $\mathbf{q}^2 \ll s$, we then have:

$$\alpha_+ \approx 1 - \frac{\mathbf{q}^2}{s}$$

$$\alpha_- \approx \frac{\mathbf{q}^2}{s}$$

We then find:

$$\alpha_- \beta_- s \approx -\frac{\mathbf{q}^4}{s} \xrightarrow{s \rightarrow \infty} 0$$

$$\alpha_+ \beta_+ s \approx 2\mathbf{q}^2 - s \xrightarrow{s \rightarrow \infty} -\infty$$

So again because of suppression, we choose the $-$ -solution and therefore obtain:

$$q^2 = -\mathbf{q}^2 - \frac{\mathbf{q}^4}{s} \quad (2.2.77)$$

Note that this implies that:

$$t \approx -\mathbf{q}^2 \quad (2.2.78)$$

Furthermore, we also have:

$$\begin{aligned} 2\mathbf{k} \cdot \mathbf{q} &= \alpha_-(\lambda_- - \rho_-)s - 2\mathbf{k} \cdot \mathbf{q} \\ &\approx \frac{\mathbf{q}^2}{s} \left(-1 + \frac{\sqrt{(M_A^2 - M_B^2 + s)^2 - 4s(\mathbf{k}^2 - M_B^2)}}{s} \right) s - 2\mathbf{k} \cdot \mathbf{q} \\ &\approx \frac{\mathbf{q}^2}{s} (M_A^2 + M_B^2 - 2\mathbf{k}^2) - 2\mathbf{k} \cdot \mathbf{q} \end{aligned}$$

So that finally, we have:

$$\begin{aligned} (k - q)^2 &= k^2 + q^2 - 2\mathbf{k} \cdot \mathbf{q} \\ &= -(\mathbf{k} - \mathbf{q})^2 - \left(\frac{M_A^2 M_B^2}{s} + \frac{\mathbf{q}^4}{s} + \frac{\mathbf{q}^2}{s} (M_A^2 + M_B^2 - 2\mathbf{k}^2) \right) \end{aligned}$$

To summarize, we have proven that:

$$k^2 \approx -\mathbf{k}^2 - \frac{M_A^2 M_B^2}{s} \equiv -\mathbf{k}^2 - m_1^2 \quad (2.2.79)$$

$$(k - q)^2 \approx -(\mathbf{k} - \mathbf{q})^2 - \left(\frac{M_A^2 M_B^2}{s} + \frac{\mathbf{q}^4}{s} + \frac{\mathbf{q}^2}{s} (M_A^2 + M_B^2 - 2\mathbf{k}^2) \right) \equiv -(\mathbf{k} - \mathbf{q})^2 - m_2^2 \quad (2.2.80)$$

* * * * *

Coming back to our sheep, we now want to get rid of the λ integrations by absorbing the deltas. While doing so, one must not forget the Jacobian factor:

$$J = \left| \begin{array}{cc} s\lambda & s\rho - s \\ s\lambda + s & s\rho \end{array} \right| = s^2 |\rho_- - \lambda_- - 1| = s^2 \frac{\sqrt{(M_A^2 - M_B^2 + s)^2 - 4s(\mathbf{k}^2 - M_B^2)}}{2s} \quad (2.2.81)$$

Now in the high-energy limit, we have:

$$J \xrightarrow{s \rightarrow \infty} s^2 \left(1 + \frac{M_A^2 + M_B^2 - \mathbf{k}^2}{s} \right) \approx s^2 \left(1 + \frac{M_A^2 + M_B^2}{s} \right) \quad (2.2.82)$$

So, in the limit we are interested in, we can take $J = s^2$ and thereby obtain:

$$\int d\Pi_{AB}^{(2)} \frac{1}{(p_1 - k_A)^2} \frac{1}{(p_1 - k_A - q)^2} = \frac{1}{8\pi^2 s} \int d^2 k \frac{1}{(\mathbf{k}^2 + m_1^2)((\mathbf{k} - \mathbf{q})^2 + m_2^2)} \quad (2.2.83)$$

which finally gives us:

$$\begin{aligned} 2\text{Im}(\mathcal{A}) &= \frac{1}{8^3\pi^4(2\pi)^2s} \int dM_A^2 \int dM_B^2 \int d^2k \frac{1}{(\mathbf{k}^2 + m_1^2)((\mathbf{k} - \mathbf{q})^2 + m_2^2)} \\ \Rightarrow \text{Im}(\mathcal{A}) &= \frac{1}{2^{12}\pi^6s} \int dM_A^2 \int dM_B^2 \int d^2k \frac{1}{(\mathbf{k}^2 + m_1^2)((\mathbf{k} - \mathbf{q})^2 + m_2^2)} \end{aligned}$$

Now let us give some integrals:

$$\int d^2\mathbf{k} \frac{1}{(\mathbf{k}^2 + m^2)^2} = \frac{\pi}{m^2} \quad (2.2.84)$$

So that if $q = 0$, we have $m_1^2 = m_2^2 = \frac{M_A^2 M_B^2}{s}$ which then gives:

$$\text{Im}(\mathcal{A}) = \frac{1}{2^{12}\pi^6s} \int dM_A^2 \int dM_B^2 \frac{\pi s}{M_A^2 M_B^2} \quad (2.2.85)$$

which is the same result that previously.

More interestingly, we have:

$$\begin{aligned} \int d^2\mathbf{k} \frac{1}{(\mathbf{k}^2 + m_1^2)((\mathbf{k} - \mathbf{q})^2 + m_2^2)} &= \int_0^1 dx \int d^2\mathbf{k} \frac{1}{[(\mathbf{k} - \mathbf{q}x)^2 + \mathbf{q}^2x(1-x) + (m_2^2 - m_1^2)x + m_1^2]^2} \\ &= (2\pi) \int_0^1 dx \int_0^\infty dk \frac{k}{[k^2 + \mathbf{q}^2x(1-x) + (m_2^2 - m_1^2)x + m_1^2]^2} \\ &= \pi \int_0^1 \frac{1}{\mathbf{q}^2x(1-x) + (m_2^2 - m_1^2)x + m_1^2} \quad (2.2.86) \end{aligned}$$

$$\begin{aligned} &= \pi \int_0^1 \frac{1}{\mathbf{q}^2x(1-x) + \frac{\mathbf{q}^2}{s}(M_A^2 + M_B^2)x + \frac{M_A^2 M_B^2}{s}} \\ &= \frac{\pi}{\mathbf{q}^2} \int_0^1 \frac{1}{x(1-x) + \frac{1}{s}(M_A^2 + M_B^2)x + \frac{M_A^2 M_B^2}{\mathbf{q}^2 s}} \quad (2.2.87) \end{aligned}$$

So that finally we have:

$$\text{Im}(\mathcal{A}) = \frac{1}{2^{12}\pi^5s\mathbf{q}^2} \int dM_A^2 \int dM_B^2 \int_0^1 \frac{1}{x(1-x) + \frac{1}{s}(M_A^2 + M_B^2)x + \frac{M_A^2 M_B^2}{\mathbf{q}^2 s}} \quad (2.2.88)$$

Now let us look at the region where $\frac{M_A^2 M_B^2}{\mathbf{q}^2 s} \gg 1$, where we end up with:

$$\text{Im}(\mathcal{A}) = \frac{1}{2^{12}\pi^5s\mathbf{q}^2} \int dM_A^2 \int dM_B^2 \frac{\mathbf{q}^2 s}{M_A^2 M_B^2}$$

where the region of integration for M_A^2, M_B^2 is the square from 0 to s , but bounded by the hyperbola $M_A^2 M_B^2 \geq \mathbf{q}^2 s$. We can do the change of variables $M_A^2 = \frac{u}{v}$ and $M_B^2 = v$ whose Jacobian is $\frac{1}{v}$ which transforms the region of integration into a triangle, so that finally we obtain:

$$\begin{aligned} \text{Im}(\mathcal{A}) &= \frac{1}{2^{12}\pi^5} \int_{\mathbf{q}^2}^s \frac{dv}{v} \int_{\mathbf{q}^2 s}^{sv} \frac{du}{u} \\ &= \frac{1}{2^{12}\pi^5} \int_{\mathbf{q}^2}^s \frac{dv}{v} \ln\left(\frac{v}{\mathbf{q}^2}\right) \\ &= \frac{1}{2^{13}\pi^5} \ln^2\left(\frac{s}{\mathbf{q}^2}\right) \end{aligned}$$

Now let us also verify that the region for which $\frac{M_A^2 M_B^2}{q^2 s} \ll 1$ is subleading. To do so, let us look more closely at our function:

$$\frac{1}{x(1-x) + \frac{1}{s}(M_A^2 + M_B^2)x + \frac{M_A^2 M_B^2}{q^2 s}} \equiv \frac{1}{x(1-x) + \alpha x + \beta} \quad (2.2.89)$$

If $\alpha = 0$, then this function is even around $\frac{1}{2}$ so that we can do the following:

$$\begin{aligned} \int_0^1 \frac{1}{x(1-x) + \beta} &= 2 \int_0^{\frac{1}{2}} \frac{1}{x(1-x) + \beta} \\ &\approx 2 \int_0^{\frac{1}{2}} \frac{1}{x + \beta} \\ &= 2 \ln \left(\frac{\frac{1}{2} + \beta}{\beta} \right) \\ &\approx -2 \ln \beta \end{aligned}$$

where in the second line, we used the fact that the the integral is dominated by the region of small x when $\beta \ll 1$ and in the last line, we used that again that $\beta \ll 1$. In particular, this result will stay true if $\alpha \ll \beta$.

On the other, if $\beta \ll \alpha$, the symmetry is broken and function is suppressed above $x \approx \frac{1}{2}$, so that we have:

$$\int_0^1 \frac{1}{x(1-x) + \alpha x + \beta} \approx -\ln \beta \quad (2.2.90)$$

For completeness sake, let us confirm this result through a Taylor expansion of the exact solution. We have:

$$\int_0^1 \frac{1}{x(1-x) + \alpha x + \beta} = \frac{\ln \left(\frac{1+\alpha+2\beta+\sqrt{(1+\alpha)^2+4\beta}}{1+\alpha+2\beta-\sqrt{(1+\alpha)^2+4\beta}} \right)}{\sqrt{(1+\alpha)^2+4\beta}} \quad (2.2.91)$$

A Taylor expansion then yields:

$$\frac{\ln \left(\frac{1+\alpha+2\beta+\sqrt{(1+\alpha)^2+4\beta}}{1+\alpha+2\beta-\sqrt{(1+\alpha)^2+4\beta}} \right)}{\sqrt{(1+\alpha)^2+4\beta}} \approx -\ln(\beta^2 + \beta\alpha) \approx \begin{cases} -2 \ln(\beta) & \text{if } \beta \gg \alpha \\ -\ln(\beta) - \ln(\alpha) \approx -\ln(\beta) & \text{if } \beta \ll \alpha \end{cases} \quad (2.2.92)$$

which confirms the previous analysis. One could now do a more precise analysis as to which of the two solutions is relevant in what domain of the $\{M_A^2, M_B^2\}$ integral, but since we are interested in the leading behavior, a factor of 2 is irrelevant to this result. We therefore end up with the integral:

$$\text{Im}(\mathcal{A}) = \frac{-1}{2^{12} \pi^5 q^2 s} \int dM_A^2 \int dM_B^2 \ln \left(\frac{M_A^2 M_B^2}{q^2 s} \right) \quad (2.2.93)$$

where region of integration is now the square from 0 to s but bounded by the hyperbola $M_A^2 M_B^2 = q^2 s$. To do, this integral we do the change of variables:

$$\frac{M_A^2}{q^2} = \frac{u}{v} \quad \& \quad \frac{M_B^2}{s} = v \quad \Rightarrow \quad |J| = \frac{q^2 s}{v} \quad (2.2.94)$$

This transforms the region of integration into a square from 0 to 1 but where the lower side is slightly inclined. We therefore obtain:

$$\begin{aligned}\text{Im}(\mathcal{A}) &= \frac{-1}{2^{12}\pi^5} \int_0^1 du \int_{\frac{q^2}{s}u}^1 dv \frac{\ln u}{v} \\ &= \frac{1}{2^{12}\pi^5} \left(2 + \ln \left(\frac{s}{q^2} \right) \right) \\ &\approx \frac{1}{2^{12}\pi^5} \ln \left(\frac{s}{q^2} \right)\end{aligned}$$

which proves that the contribution from region $\frac{M_A^2 M_B^2}{q^2 s} \ll 1$ is subleading. So we have therefore proven that in the leading logarithm approximation, we have:

$$\text{Im}(\mathcal{A}) = \frac{1}{2^{13}\pi^4} \ln^2 \left(\frac{s}{-t} \right) \quad (2.2.95)$$

Let us summarize the results we have obtained so far. The dominant region is the region for which $\frac{M_A^2 M_B^2}{q^2 s} \gg 1$ and in this region we obtain a $\ln^2$.

2.2.3.4 Leading logarithm resummation for ϕ^3 four-point function

We now present an attempt to find the result for n -rungs. We therefore consider the diagram presented in figure 2.11.

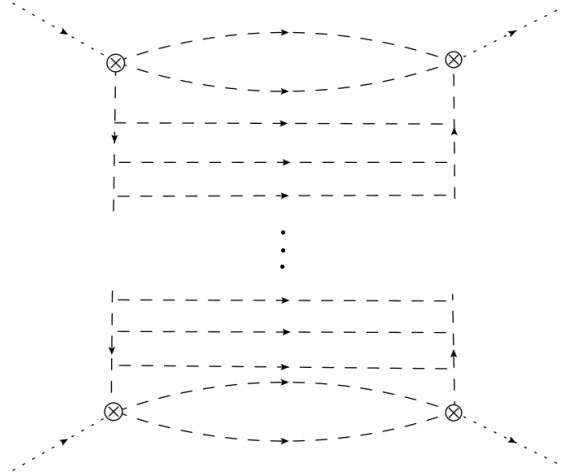


Figure 2.11: n -rung diagram for the DSA.

We start from the following point:

$$\text{Im}\mathcal{A} = \frac{1}{2^8\pi^4} \int dM_A^2 \int dM_B^2 \int d\Pi_{AB}^{(n+2)} \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger$$

where we have followed the same steps as before to replace the double massless particles by one single massive particle. We then have:

$$\begin{aligned}\int d\Pi_{AB}^{(n+2)} &= \frac{s^{n+1}}{2^{4n+3}\pi^{3n+2}} \int \prod_{i=1}^{n+1} d\rho_i d\lambda_i d^2\mathbf{k}_i \prod_{j=1}^n \delta(s(\rho_j - \rho_{j+1})(\lambda_j - \lambda_{j+1}) - (\mathbf{k}_j - \mathbf{k}_{j+1})^2 - m^2) \\ &\quad \delta(-s(1 - \rho_1)\lambda_1 - (\mathbf{k}_1^2 + M_A^2)) \delta(s(1 + \lambda_{n+1})\rho_{n+1} - (\mathbf{k}_{n+1}^2 + M_B^2))\end{aligned}$$

Imposing strong ordering hypothesis and taking $\mathbf{k}_i^2 = \mathbf{k}^2 \quad \forall i$, we obtain:

$$\int d\Pi_{AB}^{(n+2)} = \frac{s^{n+1}}{2^{4n+3}\pi^{3n+2}} \int \prod_{i=1}^{n+1} d\rho_i d\lambda_i d^2\mathbf{k}_i \prod_{j=1}^n \delta(-s\rho_j \lambda_{j+1} - \mathbf{k}^2) \\ \times \delta(-s\lambda_1 - (\mathbf{k}_1^2 + M_A^2)) \delta(s\rho_{n+1} - (\mathbf{k}_{n+1}^2 + M_B^2))$$

Rewriting the deltas, we obtain:

$$\int d\Pi_{AB}^{(n+2)} = \frac{1}{2^{4n+3}\pi^{3n+2}} \int \left(\prod_{i=1}^n \frac{d\rho_i}{\rho_i} d\lambda_i d^2\mathbf{k}_i \right) \\ \int d\rho_{n+1} d\lambda_{n+1} d^2\mathbf{k}_{n+1} \prod_{j=1}^n \delta\left(\lambda_{j+1} + \frac{\mathbf{k}^2}{s\rho_j}\right) \delta\left(\lambda_1 + \frac{(\mathbf{k}^2 + M_A^2)}{s}\right) \delta\left(s\rho_{n+1} - (\mathbf{k}^2 + M_B^2)\right)$$

We therefore have:

$$\lambda_1 = -\frac{\mathbf{k}_1^2 + M_A^2}{s} \quad \& \quad \lambda_j = -\frac{\mathbf{k}^2}{s\rho_{j-1}} \quad \forall j > 2 \\ \Rightarrow \quad \lambda_1 \rho_1 s = -(\mathbf{k}^2 + M_A^2) \rho_1 \quad \& \quad \lambda_j \rho_j s = -\mathbf{k}^2 \frac{\rho_j}{\rho_{j-1}} \quad \forall j > 2 \\ \Rightarrow \quad k_1^2 = -\mathbf{k}_1^2 - (\mathbf{k}^2 + M_A^2) \rho_1 \quad \& \quad k_j^2 = -\mathbf{k}_j^2 - \mathbf{k}^2 \frac{\rho_j}{\rho_{j-1}} \quad \forall j > 2 \quad \Rightarrow \quad k_i^2 \equiv -\mathbf{k}_i^2 - \alpha_i \tau_i \quad \forall i$$

where $\alpha_i = \mathbf{k}^2 \quad \forall i \neq 1$ and $\alpha_1 = \mathbf{k}^2 + M_A^2$ and $\tau_i = \frac{\rho_j}{\rho_{j-1}}$. Absorbing the delta functions then leaves us with:

$$\int d\Pi_{AB}^{(n+2)} \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger = \frac{1}{8\pi^2 s} \left[\frac{g^{2n}}{2^{4n}\pi^{3n}} \prod_{j=1}^{n+1} \int d^2\mathbf{k}_j \prod_{i=1}^n \int_{\rho_{i+1}}^1 \frac{d\rho_i}{\rho_i} \int_0^1 d\rho_{n+1} \right. \\ \left. \times \prod_{m=1}^{n+1} \frac{1}{(\mathbf{k}_m^2 + m^2 + \alpha_m \tau_m)((\mathbf{k}_m - \mathbf{q})^2 + m^2 + \alpha_m \tau_m)} s \delta\left(s\rho_{n+1} - (\mathbf{k}^2 + M_B^2)\right) \right]$$

Doing the Mellin transform with respect to $\frac{s}{\mathbf{k}^2 + M_B^2}$ then yields:

$$\frac{1}{8\pi^2 s} \mathcal{MF} \left[8\pi^2 s \int d\Pi_{AB}^{(n+2)} \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger \right] = \frac{1}{8\pi^2 s} \left(\frac{g^2}{16\pi^3} \right)^n \prod_{j=1}^{n+1} \int d^2\mathbf{k}_j \prod_{i=1}^n \int_{\rho_{i+1}}^1 \frac{d\rho_i}{\rho_i} \int_0^1 \frac{d\rho_{n+1}}{\rho_{n+1}} \rho_{n+1}^\omega \\ \times \prod_{m=1}^{n+1} \frac{1}{(\mathbf{k}_m^2 + m^2 + \alpha_m \tau_m)((\mathbf{k}_m - \mathbf{q})^2 + m^2 + \alpha_m \tau_m)}$$

Finally, we can do the change of variables to the τ_i s to entangle all the integrals and we

obtain:

$$\begin{aligned}
& \frac{1}{8\pi^2 s} \mathcal{MF} \left[8\pi^2 s \int d\Pi_{AB}^{(n+2)} \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger \right] \\
&= \frac{1}{8\pi^2 s} \left(\frac{g^2}{16\pi^3} \right)^n \prod_{j=1}^n \int d^2 \mathbf{k}_j \prod_{i=1}^{n+1} \int_0^1 \frac{d\tau_i \tau_i^{\omega-1}}{(\mathbf{k}_i^2 + m^2 + \alpha_i \tau_i)((\mathbf{k}_i - \mathbf{q})^2 + m^2 + \alpha_i \tau_i)} \\
&= \frac{1}{8\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{16\pi^3 \mathbf{q}^2} \right)^n \prod_{i=1}^{n+1} \left(\int_0^1 d\tau_i \int_0^1 dx_i \frac{\tau_i^{\omega-1}}{x_i(1-x_i) + \frac{m^2}{\mathbf{q}^2} + \frac{\alpha_i}{\mathbf{q}^2} \tau_i} \right) \\
&= \frac{1}{8\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{16\pi^3 \mathbf{q}^2} \right)^n \prod_{i=1}^{n+1} \left(\int_0^1 d\tau_i \int_0^{\frac{1}{2}} dx_i \frac{2\tau_i^{\omega-1}}{x_i(1-x_i) + \frac{m^2}{\mathbf{q}^2} + \frac{\alpha_i}{\mathbf{q}^2} \tau_i} \right) \\
&= \frac{1}{8\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{16\pi^3 \mathbf{q}^2} \right)^n \prod_{i=2}^{n+1} \left(\frac{2}{\omega} \int_0^{\frac{1}{2}} dx_i \frac{1}{x_i(1-x_i) + \frac{m^2}{\mathbf{q}^2}} \right) \int_0^{\frac{1}{2}} dx_1 \int_0^1 d\tau_1 \frac{2\tau_1^{\omega-1}}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2} + \frac{M_A^2}{\mathbf{q}^2} \tau_1} \\
&= \frac{1}{4\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\frac{1}{\omega} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \\
&\quad \times \frac{1}{\omega} \int_0^{\frac{1}{2}} dx_1 \frac{1}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2}} {}_2F_1 \left(1, \omega, 1 + \omega, -\frac{\frac{M_A^2}{\mathbf{q}^2}}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2}} \right) \\
&= \frac{1}{4\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \sum_{k=0}^{\infty} \frac{1}{\omega^n} \frac{1}{\omega + k} (-1)^k \int_0^{\frac{1}{2}} dx_1 \frac{\left(\frac{M_A^2}{\mathbf{q}^2} \right)^k}{\left(x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2} \right)^{k+1}}
\end{aligned}$$

We need to find the residue of of the following expressions:

$$f_k(\omega) \equiv \frac{1}{\omega^n} \frac{1}{\omega + k} \left(\frac{s}{\mathbf{k}'^2} \right)^\omega = \frac{1}{\omega^n} \frac{1}{\omega + k} \exp \left(\omega \ln \left(\frac{s}{\mathbf{k}'^2} \right) \right)$$

If $k = 0$, we have:

$$\text{Res}(f_0(\omega), 0) = \frac{1}{n!} \ln \left(\frac{s}{\mathbf{k}'^2} \right)^n$$

If $k \neq 0$, we have:

$$\begin{aligned}
\text{Res}(f_k(\omega), 0) &= (-1)^{n-1} \sum_{l=0}^{n-1} \frac{(-1)^l \ln \left(\frac{s}{\mathbf{k}'^2} \right)^l}{l!} \frac{1}{k^{n-l}} \\
\text{Res}(f_k(\omega), -k) &= \frac{(-1)^n}{k^n} \left(\frac{\mathbf{k}'^2}{s} \right)^k
\end{aligned}$$

So that finally after taking the inverse Mellin transform, we have:

$$\begin{aligned}
\int d\Pi_{AB}^{(n+2)} \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger &= \frac{1}{4\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \\
&\times \int_0^{\frac{1}{2}} dx_1 \left[\frac{1}{n!} \ln \left(\frac{s}{\mathbf{k}'^2} \right)^n + \sum_{k=1}^{\infty} \left((-1)^{k+n-1} \frac{\left(\frac{M_A^2}{\mathbf{q}^2} \right)^k}{\left(x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2} \right)^{k+1}} \left(\sum_{l=0}^{n-1} \frac{(-1)^l \ln \left(\frac{s}{\mathbf{k}'^2} \right)^l}{l!} \frac{1}{k^{n-l}} - \frac{1}{k^n} \left(\frac{\mathbf{k}'^2}{s} \right)^k \right) \right]
\end{aligned}$$

Let us focus on the last term:

$$\begin{aligned}
\int d\Pi_{AB}^{(n+2)} \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger &= \frac{1}{4\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \\
&\quad \times \int_0^{\frac{1}{2}} dx_1 \left[(-1)^n \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)^n} \left(1 + \frac{1}{k} \right)^n \frac{\left(\frac{M_A^2 M_B^2}{\mathbf{q}^2 s} \right)^k}{\left(x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2} \right)^{k+1}} \right] \\
&\approx \frac{1}{4\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \int_0^{\frac{1}{2}} dx_1 \left[\frac{(-1)^{n+1}}{\frac{M_A^2 M_B^2}{\mathbf{q}^2 s}} \sum_{k=2}^{\infty} \frac{(-1)^k}{k^n} \frac{\left(\frac{M_A^2 M_B^2}{\mathbf{q}^2 s} \right)^k}{\left(x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2} \right)^k} \right] \\
&= \frac{1}{4\pi^2 s \mathbf{q}^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \\
&\quad \times \int_0^{\frac{1}{2}} dx_1 \left[\frac{(-1)^{n+1}}{\frac{M_A^2 M_B^2}{\mathbf{q}^2 s}} \left(\text{Li}_n \left(\frac{-\frac{M_A^2 M_B^2}{\mathbf{q}^2 s}}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2}} \right) - \left(-\frac{\frac{M_A^2 M_B^2}{\mathbf{q}^2 s}}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2}} \right) \right) \right] \\
&\approx \frac{1}{4\pi^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} dx_1 \frac{1}{M_A^2 M_B^2} \ln \left(\frac{\frac{M_A^2 M_B^2}{\mathbf{q}^2 s}}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2}} \right)^n
\end{aligned}$$

In the second equality, we used the fact that since $k \geq 1$, the higher powers of $\frac{1}{k}$ are subleading. In the second to last line, we also neglected the second term. All this is done for $\frac{M_A^2 M_B^2}{\mathbf{q}^2 s} \gg 1$, so we obtain:

$$\begin{aligned}
\int dM_A^2 \int dM_B^2 \int d\Pi_{AB}^{(n+2)} \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger &= \frac{1}{4\pi^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \frac{(-1)^n}{n!} \\
&\quad \times \int_0^{\frac{1}{2}} dx_1 \int dM_A^2 \int dM_B^2 \frac{1}{M_A^2 M_B^2} \ln \left(\frac{\frac{M_A^2 M_B^2}{\mathbf{q}^2 s}}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2}} \right)^n \\
&= \frac{1}{4\pi^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} dx_1 \int_{\mathbf{q}^2}^s \frac{dv}{v} \int_{\mathbf{q}^2 s}^{sv} \frac{du}{u} \ln \left(\frac{\frac{u}{\mathbf{q}^2 s}}{x_1(1-x_1) + \frac{m^2}{\mathbf{q}^2}} \right)^n \\
&= \frac{1}{4\pi^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} dx_1 \int_{\mathbf{q}^2}^s \frac{dv}{v} \int_{\mathbf{q}^2 s}^{sv} \frac{du}{u} \left[\ln \left(\frac{u}{\mathbf{q}^2 s} \right)^n + (\dots) \right] \\
&= \frac{1}{4\pi^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} dx_1 \frac{1}{(n+1)(n+2)} \ln \left(\frac{s}{\mathbf{q}^2} \right)^{n+2} + (\dots) \\
&= \frac{1}{8\pi^2} \left(\frac{g^2}{8\pi^3 \mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \frac{(-1)^{n+2}}{(n+2)!} \ln \left(\frac{s}{\mathbf{q}^2} \right)^{n+2} + (\dots)
\end{aligned}$$

So that finally, we have:

$$\begin{aligned}
\text{Im}\mathcal{A} &= \frac{1}{2^{11}\pi^6} \left(\frac{g^2}{8\pi^3\mathbf{q}^2} \right)^n \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^n \frac{(-1)^{n+2}}{(n+2)!} \ln \left(\frac{s}{\mathbf{q}^2} \right)^{n+2} + (\dots) \quad (2.2.96) \\
&= \frac{1}{2^{11}\pi^6} \frac{1}{\left(\frac{g^2}{8\pi^3\mathbf{q}^2} \right)^2 \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^2} \\
&\quad \times \frac{(-1)^{n+2}}{(n+2)!} \left(\frac{g^2}{8\pi^3\mathbf{q}^2} \right)^{n+2} \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^{n+2} \ln \left(\frac{s}{\mathbf{q}^2} \right)^{n+2} + (\dots) \quad (2.2.97)
\end{aligned}$$

If we were to resum, we obtain:

$$\text{Im}\mathcal{A} = \frac{1}{2^{11}\pi^6} \frac{1}{\left(\frac{g^2}{8\pi^3\mathbf{q}^2} \right)^2 \left(\int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \right)^2} \left(\left(\frac{s}{\mathbf{q}^2} \right)^{-\alpha} + \alpha \ln \left(\frac{s}{\mathbf{q}^2} \right) - 1 \right) + (\dots) \quad (2.2.98)$$

where

$$\alpha = \frac{g^2}{8\pi^3\mathbf{q}^2} \int_0^{\frac{1}{2}} dx \frac{1}{x(1-x) + \frac{m^2}{\mathbf{q}^2}} \geq 0 \quad (2.2.99)$$

A comment is order here: The fact that the amplitude is proportional to g^{-4} does not a priori invalidate our argument. We remind the reader that in this section $\mathcal{A}$ does not stand for the DSA. Indeed to relate the above result to the DSA, we would need to multiply by four beta functions which all contain at least one power of g , thereby canceling the g in the denominator.

2.2.3.4.1 Discussion The above computation certainly warrants a discussion. Let us start by saying that we do believe that the result is correct, or at least plausible. The reason is the following. For scalar fields, there exists a more straightforward way of dealing with ladder diagrams. We will only present the ideas of how to proceed. For more details, see [19]. Very generally, the standard integral for a loop diagram can be rewritten using the technique of Feynman parameters as:

$$\int_0^1 \prod_{r=1}^n d\alpha_r \frac{\delta(\sum \alpha - 1) C^{n-2l-2}}{[D + i\epsilon]^{n-2l}} \quad (2.2.100)$$

Explaining what the various symbols represent would take us to far, the main point we want to make is that there exists a technique akin to cutting rules to determine C and D . In particular, this technique very quickly implies that for a ladder diagram the above integral takes the form

$$\int_0^1 \prod d\alpha_r \frac{\delta(\sum \alpha - 1) N(\alpha)}{[g(\alpha)s + d(t, \alpha)]^n} \quad (2.2.101)$$

One can then argue that the dominant contribution comes from the corner at the origin of the hypercube over which the Feynman parameters are integrated and that this contribution will be proportional to $\ln^{n-1} s$. Now, we believe, without having investigated this in any detail, that this argument can be readily adapted to the dilaton case where

there are simply four more lines to be cut (in the sense mentioned previously) yielding four more powers of logarithms for a total of $\ln^{n+3} s$. If the amplitude goes as $\ln^{n+3} s$, then its imaginary part essentially goes as $\ln^{n+2} s$, which is the result we obtained. We do emphasize that this for now is a purely heuristic argument, but which we believe worth investigating in the future.

On the other hand, there are quite a number of questionable steps in computation presented in this section. First of all, we used the hypothesis that the dominant contribution came from the region where we could make the approximation:

$$\mathbf{k}_i^2 = (\mathbf{k}_i - \mathbf{k}_{i+1})^2 = \mathbf{k}^2 \quad (2.2.102)$$

where $\mathbf{k}^2$ is order $|t|$. This could be justified a posteriori in the original case because of the denominator structure. It is not all clear whether this remains true in the dilaton case. In the course of the calculation, we have also neglected a good deal of terms, which at first naive look certainly do not seem to be subleading. A related problem is that given how we introduced the masses M_A and M_B , we expect a priori the result before integration to be symmetric under their exchange. One problem with this is that the original method is inherently not symmetric because it treats the first and last rungs very differently. There is symmetric way of proceeding, but we could not get far following that path. This symmetry could also be a blessing in disguise. If we could for instance argue that any non-symmetric contribution should not matter, a good of the problematic terms would disappear. We have started some explorations into this, but we do not present them in this thesis because they are very far from completion.

We finally want to mention that even if the above computation were to be correct, we would not be out of the woods yet. Indeed, we remind the reader that the DSA is a combination of 2-,3- and 4-point functions of T and we would therefore still have to compute the 2- and 3-point functions. Here, we arrive at another problem, which is that it does not seem like there is a straightforward adaptation of the presented methods to the 3-point function, but this question has not been explored yet. This is also the reason why we did not study ϕ^4 , which would have been a more logical choice than ϕ^3 since ϕ^4 is marginal in four dimensions. Let us finally point out that in the forward limit $t \rightarrow 0$, our result is well defined and vanishing. Trying to make any general conclusions on the behavior of the DSA in the forward limit would be premature.

As a conclusion, we can certainly say that the above computation is very questionable and might very well be wrong. We do believe that the result is plausible, but the steps of the computation need much justification.

2.3 Solution avenues for the on-shell limit

This section does not contain any formal developments. It only mentions two ideas of solutions, which have not yet been explored in any significant way.

We mentioned in a previous section that at order B^2 the corresponding two-point functions are well defined in the on-shell limit. To check whether the amplitude is well defined to order B^3 , one would need to check the properties of three-point function of conformal theories in momentum space. To do so, the idea would be to use the results from [20]. This should be a rather straightforward application and from a quick look, we do not expect any complications at this level.

The other idea to check whether the on-shell limit is well-defined is to do a twist expansion OPE and then taking the external legs to be on the light-cone. Explorations into this have been started[21].

Conclusion

This thesis was separated into two chapters, from each of which one should get a different use. The first chapter is a (hopefully) comprehensive presentation of the techniques involved in the proof of the a-theorem both in two and in four dimensions. We presented the local RG equation and the constraints it imposes on QFTs and how, when combined with techniques related to the dilaton, it becomes a powerful tool to study some properties of the RG flow. In particular, in the first section, we have presented a thorough discussion of how these techniques can be applied in two dimensions, thereby giving a “new” proof for some of the classic results of two dimensional CFTs. In particular, we give a version of the non-perturbative proof of the c-theorem, which had not been written down in all detail up until now.

The main goal of the second chapter was to convince the reader that there are indeed some potential problems in the proof of a-theorem. We presented these problems which essentially boil down to whether or not the the dilaton scattering amplitude is well defined, and argued why the resolution of these problems is not trivial and certainly worth investigating. We then presented some ideas of how to start trying to deal with these. We started, without finding any definite results, exploring the option of computing explicitly the dilaton scattering amplitude in a simple theory, so as to check its well-definiteness.

There are several paths which can be followed from here on out. The first is to complete the computation in ϕ^3 and then use the experience gained from this computation to compute the dilaton scattering amplitude in the much more interesting and relevant case of pure Yang-Mills theory. The second path is to study the on-shell condition for 3- and 4-point functions of CFTs. As already mentioned, the case of 3-point functions should be a straightforward application of the paper by Bzowski, McFadden and Skenderis ([20]). The case of four-point functions would probably be much less straightforward. One could for instance try to generalize the results of [20] and study the 4-point functions of CFTs in momentum space to explore the consequences the on-shell condition has on the latter. Finally, one could also explore whether it is possible to find any positivity constraints on the dilaton scattering amplitude without having to take the potentially problematic forward limit.

Appendix A

Results in Conformal Field Theory

A.1 Two dimensional CFT

This section follows [22] and [23]. Let us work in an Euclidean flat metric. The Lorentzian signature case can be addressed in very much the same way. One can then introduce complex variables:

$$z = x^0 + ix^1 \quad \& \quad \bar{z} = x^0 - ix^1 \quad (\text{A.1.1})$$

and the corresponding holomorphic derivatives:

$$\partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \quad \& \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \quad (\text{A.1.2})$$

One can then compute the metric in the new coordinates. For instance:

$$g_{zz} = \frac{\partial x^i}{\partial z} \frac{\partial x^j}{\partial z} \delta_{ij} = \frac{1}{2} - \frac{1}{2} = 0 \quad (\text{A.1.3})$$

The result is that:

$$g_{ab} = \frac{1}{2}J \quad \& \quad g^{ab} = 2J \quad (\text{A.1.4})$$

where

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{A.1.5})$$

So that the line element becomes:

$$ds^2 = dzd\bar{z} \quad (\text{A.1.6})$$

Let us note that most of the time, it is convenient to treat z and $\bar{z}$ as independent variables, which means that we really considering $\mathbb{C}^2$ instead of $\mathbb{R}^2$, but one should not forget that ultimately, physics is sitting on the real slice $\mathbb{R}^2 \subset \mathbb{C}^2$ defined by $\bar{z} = z^*$.

Let us simply mention that in complex coordinates we have:

$$T_\alpha^\alpha = 4T_{z\bar{z}} \quad (\text{A.1.7})$$

$$\partial_a T^{ab} = \partial_z T^{zz} + \partial_{\bar{z}} T^{\bar{z}z} \quad \& \quad \partial_z T^{z\bar{z}} + \partial_{\bar{z}} T^{\bar{z}\bar{z}} \quad (\text{A.1.8})$$

It is also easy to show that in two dimensions conformal transformations are simply given by holomorphic functions. Indeed, we have:

$$z \rightarrow z' = f(z) \quad \text{and} \quad \bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z}) \quad \Rightarrow \quad ds^2 \rightarrow \left| \frac{df}{dz} \right|^2 ds^2 \quad (\text{A.1.9})$$

A.2 Two-point functions in CFTs and unitarity bounds

We here only cite two basic results about CFTs. The first one concerns the form of correlators in CFTs. One can prove that scalar two-point functions are given by (see [24]):

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \frac{c_{12} \delta_{\Delta_1 \Delta_2}}{(x_1 - x_2)^{2\Delta_1}} \quad (\text{A.2.1})$$

The second result is that unitarity gives constraints on the conformal dimension of primary operators. In four dimensions, we have that the conformal dimension of primary operators with a (j_1, j_2) representation under the Lorentz group must satisfy (see [24])

$$\Delta \geq j_1 + j_2 + 2 - \delta_{j_1 \cdot j_2, 0} \quad (\text{A.2.2})$$

A.3 Weyl anomaly in four dimensions

Following the same procedure we did in section 1.2.1.3 where we imposed the conditions:

1. We take $\mathcal{A}_\sigma$ to be a local functional of the metric.
2. $\mathcal{A}_\sigma$ has to be diffeomorphism invariant.
3. $\mathcal{A}_\sigma$ has to be linear in σ .
4. $\mathcal{A}_\sigma$ has to be of dimension D .
5. $\mathcal{A}_\sigma$ has to satisfy the WZ consistency condition:

$$\Delta_{\sigma_1} \int d^D x \mathcal{A}_{\sigma_2}[\mathcal{J}] = \Delta_{\sigma_2} \int d^D x \mathcal{A}_{\sigma_1}[\mathcal{J}] \quad (\text{A.3.1})$$

In 4 dimensions, the terms that satisfy the conditions are:

1. $\sigma \hat{R}_{\mu\nu\lambda\rho} \hat{R}^{\mu\nu\lambda\rho}$
2. $\sigma \hat{R}_{\mu\nu} \hat{R}^{\mu\nu}$
3. $\sigma \hat{R}^2$
4. $\sigma \square \hat{R}$
5. $\sigma \varepsilon^{\mu\nu\lambda\rho} \hat{R}_{\mu\nu\alpha\beta} \hat{R}^{\alpha\beta}_{\lambda\rho} \equiv \sigma S_4$
6. $\sigma \Lambda^4$

It turns out that there is a more useful basis for the first three terms, namely:

1. $\hat{R}^2$
2. $\hat{E}_4 = \hat{R}_{\mu\nu\lambda\rho} \hat{R}^{\mu\nu\lambda\rho} - 4 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \hat{R}^2$
3. $\hat{W}_{\mu\nu\lambda\rho} \hat{W}^{\mu\nu\lambda\rho} = \hat{R}_{\mu\nu\lambda\rho} \hat{R}^{\mu\nu\lambda\rho} - 2 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \frac{1}{3} \hat{R}^2$

where $\hat{E}_4$ is the 4 dimensional Euler density and $\hat{W}_{\mu\nu\lambda\rho}$ is the Weyl tensor.

The Weyl tensor is constructed so as to be Weyl invariant. In 4 dimensions, the variation of R is given by:

$$\begin{aligned}\tilde{\hat{R}} &= \Omega^{-2} \hat{R} - 6\Omega^{-3} (\partial_\alpha \partial_\beta \Omega) g^{\alpha\beta} \\ &= e^{2\sigma} (\hat{R} + 6\Box\sigma) \\ &\approx \hat{R} + 2\sigma\hat{R} + 6\Box\sigma\end{aligned}$$

We then have that $\hat{R}^2$ transforms under Weyl transformations as:

$$\begin{aligned}\Delta_{\sigma_2}^W \int d^4x \sqrt{-\hat{g}(x)} \sigma_1(x) \hat{R}^2 &= \int d^4x \left[-4\sigma_2(x) \sqrt{-\hat{g}(x)} \sigma_1(x) \hat{R}^2 + \sqrt{-\hat{g}(x)} \sigma_1(x) 2\hat{R} (2\sigma_2\hat{R} + 6\Box\sigma_2) \right] \\ &= +12 \int d^4x \sqrt{-\hat{g}(x)} \hat{R}(x) \sigma_1(x) \Box\sigma_2(x)\end{aligned}$$

This proves that $\hat{R}^2$ does not satisfy

$$\Delta_{\sigma_2}^W \int d^4x \sqrt{-\hat{g}(x)} \sigma_1(x) \hat{R}^2 = \Delta_{\sigma_1}^W \int d^4x \sqrt{-\hat{g}(x)} \sigma_2(x) \hat{R}^2$$

because when doing the integration by parts the Ricci scalar will get caught in the middle. It is too early to conclude that we can leave $\hat{R}^2$ off the list of anomalies, since it could a priori be that its contribution is canceled by the variations of one of the other terms.

The previous calculation also shows the following:

$$\Delta_\sigma^W \left(\frac{1}{12} \int d^4x \sqrt{-\hat{g}(x)} \hat{R}^2(x) \right) = \int d^4x \sqrt{-\hat{g}(x)} \Box\sigma(x) \hat{R}(x) = \int d^4x \sqrt{-\hat{g}(x)} \sigma(x) \Box\hat{R}(x) \quad (\text{A.3.2})$$

We can then define a new generating functional by adding this local counterterm:

$$\tilde{\mathcal{W}}[\hat{g}] = \mathcal{W}[\hat{g}] - \frac{1}{12} \int d^4x \sqrt{-\hat{g}(x)} \hat{R}^2(x)$$

thereby eliminating the $\sigma\Box\hat{R}$ term in the list of possible anomalies.

The last piece we are missing is the transformation of $\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}$ under Weyl transformations:

$$\begin{aligned}\tilde{\hat{R}}_{\sigma\nu} &= \hat{R}_{\sigma\nu} - [2\Omega^{-1}\partial_\sigma\partial_\nu\Omega + g_{\sigma\nu}\Omega^{-1}\Box\Omega] + 4[\Omega^{-2}\partial_\sigma\Omega\partial_\nu\Omega - g_{\sigma\nu}\Omega^{-2}(\partial\Omega)^2] \\ &\approx \hat{R}_{\sigma\nu} + [2\partial_\sigma\partial_\nu\sigma + g_{\sigma\nu}\Box\sigma] \\ \Rightarrow \tilde{\hat{R}}_{\sigma\nu}\tilde{\hat{R}}^{\sigma\nu} &\approx (1+2\sigma)^2 g^{\alpha\beta} g^{\mu\nu} \hat{R}_{\sigma\nu} \hat{R}^{\sigma\nu} \\ &\approx g^{\alpha\beta} g^{\mu\nu} \left(\hat{R}_{\alpha\mu} \hat{R}_{\beta\nu} + 2\partial_\alpha\partial_\mu\sigma \hat{R}_{\beta\nu} + g_{\alpha\mu}\Box\sigma \hat{R}_{\beta\nu} + 4\sigma \hat{R}_{\alpha\mu} \hat{R}_{\beta\nu} - 2\hat{R}_{\alpha\mu}\partial_\beta\partial_\nu\sigma + \hat{R}_{\alpha\mu}g_{\beta\nu}\Box\sigma \right) \\ \Rightarrow \delta_\sigma \left(\hat{R}_{\sigma\nu} \hat{R}^{\sigma\nu} \right) &= 2\partial^\mu\partial^\nu\sigma \hat{R}_{\mu\nu} + \Box\sigma \hat{R} + 4\hat{R}_{\mu\nu} \hat{R}^{\mu\nu}\sigma + 2\partial^\mu\partial^\nu\sigma \hat{R}_{\mu\nu} + \Box\sigma \hat{R} \\ &= +4\hat{R}_{\mu\nu} \hat{R}^{\mu\nu}\sigma + 4\partial^\mu\partial^\nu\sigma \hat{R}_{\mu\nu} + 2\Box\sigma \hat{R}\end{aligned}$$

So that finally, we obtain:

$$\begin{aligned}\Delta_{\sigma_2}^W \int d^4x \sqrt{-\hat{g}(x)} \sigma_1(x) \hat{R}_{\mu\nu}(x) \hat{R}^{\mu\nu}(x) \\ &= \int d^4x \left[-4\sigma_2(x) \sqrt{-\hat{g}(x)} \sigma_1(x) \hat{R}_{\mu\nu}(x) \hat{R}^{\mu\nu}(x) + \left(4\hat{R}_{\mu\nu} \hat{R}^{\mu\nu} \sigma_1 \sigma_2 + 4\partial^\mu\partial^\nu\sigma_2 \hat{R}_{\mu\nu} \sigma_1 + 2\sigma_1 \Box\sigma_2 \hat{R} \right) \right] \\ &= \int d^4x \sqrt{-\hat{g}(x)} \left[4\sigma_1 \partial^\mu\partial^\nu\sigma_2 \hat{R}_{\mu\nu} + 2\sigma_1 \Box\sigma_2 \hat{R} \right]\end{aligned}$$

Now, we have that:

$$\Delta_{\sigma_2}^W \int d^4x \sqrt{-\hat{g}(x)} \sigma_1(x) \hat{W}^2(x) = \int d^4x \sqrt{-\hat{g}(x)} \sigma_1(x) \sigma_2(x) \hat{W}^2(x) \quad (\text{A.3.3})$$

which is clearly symmetric under the exchange of $\sigma_1 \leftrightarrow \sigma_2$ and therefore satisfies the Wess-Zumino consistency condition. For the Euler density, we rewrite it as $\hat{E}_4 = \hat{W}^2 - 2\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + \frac{2}{3}\hat{R}^2$. We have just shown that the $\hat{W}^2$ satisfies the Wess-Zumino consistency condition, so that we can focus on the two other terms. We then have:

$$\begin{aligned} \Delta_{\sigma_2}^W \int d^4x \sqrt{-\hat{g}(x)} \sigma_1(x) \left(2\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} - \frac{2}{3}\hat{R}^2 \right) \\ = - \int d^4x \sqrt{-\hat{g}(x)} \left[-8\sigma_1 \nabla^\mu \nabla^\nu \sigma_2 \hat{R}_{\mu\nu} - 4\sigma_1 \square \sigma_2 \hat{R} + 8\hat{R} \sigma_1 \square \sigma_2 \right] \\ = - \int d^4x \sqrt{-\hat{g}(x)} \left[8\nabla^\mu \sigma_1 \nabla^\nu \sigma_2 \hat{R}_{\mu\nu} + 8\sigma_1 \nabla^\nu \sigma_2 \nabla^\mu \hat{R}_{\mu\nu} + 4\hat{R} \sigma_1 \square \sigma_2 \right] \\ = - \int d^4x \sqrt{-\hat{g}(x)} \left[8\nabla^\mu \sigma_1 \nabla^\nu \sigma_2 \hat{R}_{\mu\nu} + 4\sigma_1 \nabla^\nu \sigma_2 \nabla_\nu \hat{R} + 4\hat{R} \sigma_1 \square \sigma_2 \right] \\ = - \int d^4x \sqrt{-\hat{g}(x)} \left[8\nabla^\mu \sigma_1 \nabla^\nu \sigma_2 \hat{R}_{\mu\nu} - 4\nabla_\nu \sigma_1 \nabla^\nu \sigma_2 \hat{R} - 4\sigma_1 \square \sigma_2 \hat{R} + 4\hat{R} \sigma_1 \square \sigma_2 \right] \\ = \int d^4x \sqrt{-\hat{g}(x)} \left[4\nabla_\nu \sigma_1 \nabla^\nu \sigma_2 \hat{R} - 8\nabla^\mu \sigma_1 \nabla^\nu \sigma_2 \hat{R}_{\mu\nu} \right] \end{aligned}$$

Notice that we have left out in the above calculation the variation of $\sqrt{-\hat{g}(x)}$ because it is trivially symmetric in $\sigma_1 \leftrightarrow \sigma_2$ as pointed out before. The final expression is now manifestly symmetric under the exchange of $\sigma_1 \leftrightarrow \sigma_2$. We have therefore proven that E_4 satisfies the Wess-Zumino consistency condition. We also see that in this basis $\hat{E}_4$ and $\hat{W}^2$ satisfy the Wess-Zumino consistency conditions on their own, so that now can eliminate the $\hat{R}^2$ anomaly. So we finally have ended up with:

$$\Delta_\sigma^W \mathcal{W}[\hat{g}] = \int d^4x \sqrt{-\hat{g}(x)} \left[\sigma \left(-c_4 \hat{W}^2 + a_4 \hat{E}_4 + e_4 \hat{S}_4 + f_4 \Lambda^4 \right) \right] \quad (\text{A.3.4})$$

We therefore have:

$$\langle T(x) \rangle = -c_4 \hat{W}^2 + a_4 \hat{E}_4 + e_4 \hat{S}_4 + f_4 \Lambda^4 \quad (\text{A.3.5})$$

For a parity even theory, we can eliminate the last term and obtain:

$$\langle T(x) \rangle = a_4 \hat{E}_4(x) - c_4 \hat{W}^2(x) + f_4 \Lambda^4 \quad (\text{A.3.6})$$

Appendix B

Games with “cutting rules”

B.1 Position space “cutting rules” for operator correlators

Let $\mathcal{O}$ be a Hermitian operator and for simplicity let $\mathcal{O}$ also be bosonic. Let us study the following quantity:

$$\begin{aligned}\tilde{I}^{(4)} &\equiv \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \exp(ip_k x_k) 2\text{Re} [\langle \mathbf{T}\{\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\} \rangle] \\ &= \int d\mathbf{x} \exp(ip_k x_k) (\langle \mathbf{T}\{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\} \rangle + \langle \mathbf{T}\{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\} \rangle^*)\end{aligned}$$

By definition, we have:

$$\begin{aligned}\langle \mathbf{T}\{\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\} \rangle \\ = \sum_{\pi \in S(4)} \theta(t_{\pi(1)} - t_{\pi(2)})\theta(t_{\pi(2)} - t_{\pi(3)})\theta(t_{\pi(3)} - t_{\pi(4)}) \langle \mathcal{O}(x_{\pi(1)})\mathcal{O}(x_{\pi(2)})\mathcal{O}(x_{\pi(3)})\mathcal{O}(x_{\pi(4)}) \rangle\end{aligned}$$

Let us write collectively:

$$d^4x_1 d^4x_2 d^4x_3 d^4x_4 \equiv d\mathbf{x}$$

The first term then can be rewritten as follows:

$$\begin{aligned}\sum_{\pi \in S(4)} \int d\mathbf{x} \exp(ip_k x_k) \theta(t_{\pi(1)} - t_{\pi(2)})\theta(t_{\pi(2)} - t_{\pi(3)})\theta(t_{\pi(3)} - t_{\pi(4)}) \langle \mathcal{O}(x_{\pi(1)})\mathcal{O}(x_{\pi(2)})\mathcal{O}(x_{\pi(3)})\mathcal{O}(x_{\pi(4)}) \rangle \\ = \sum_{\pi \in S(4)} \int d\tilde{\mathbf{x}} \exp(ip_k x_{\pi^{-1}(k)}) \theta(\tilde{t}_1 - \tilde{t}_2)\theta(\tilde{t}_2 - \tilde{t}_3)\theta(\tilde{t}_3 - \tilde{t}_4) \langle \mathcal{O}(\tilde{x}_1)\mathcal{O}(\tilde{x}_2)\mathcal{O}(\tilde{x}_3)\mathcal{O}(\tilde{x}_4) \rangle \\ = \sum_{\pi \in S(4)} \int d\mathbf{x} \exp(ip_{\pi(k)} x_k) \theta(t_1 - t_2)\theta(t_2 - t_3)\theta(t_3 - t_4) \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle\end{aligned}$$

where in the second line, we made the change of variables $\tilde{x}_i = x_{\pi(i)}$ and in the last line, we rewrote $p_k x_{\pi^{-1}(k)} = p_{\pi(k)} \tilde{x}_k$ and renamed the $\tilde{x}$ to x to avoid clutter.

For the second term, the same rewritting yields:

$$\begin{aligned}
 & \sum_{\pi \in S(4)} \int \mathbf{d}x \exp(ip_k x_k) \theta(t_{\pi(1)} - t_{\pi(2)}) \theta(t_{\pi(2)} - t_{\pi(3)}) \theta(t_{\pi(3)} - t_{\pi(4)}) \langle \mathcal{O}(x_{\pi(1)}) \mathcal{O}(x_{\pi(2)}) \mathcal{O}(x_{\pi(3)}) \mathcal{O}(x_{\pi(4)}) \rangle^* \\
 &= \sum_{\pi \in S(4)} \int \mathbf{d}x \exp(ip_k x_k) \theta(t_{\pi(1)} - t_{\pi(2)}) \theta(t_{\pi(2)} - t_{\pi(3)}) \theta(t_{\pi(3)} - t_{\pi(4)}) \\
 &\quad \times \langle \mathcal{O}(x_{\pi(4)}) \mathcal{O}(x_{\pi(3)}) \mathcal{O}(x_{\pi(2)}) \mathcal{O}(x_{\pi(1)}) \rangle \\
 &= \sum_{\pi \in S(4)} \int \mathbf{d}x \exp(ip_{\pi(k)} x_k) \theta(t_4 - t_3) \theta(t_3 - t_2) \theta(t_2 - t_1) \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle
 \end{aligned}$$

where in the last line, we have made the change of variables backwards, meaning $\tilde{x}_1 = x_{\pi(4)}$ until $\tilde{x}_4 = x_{\pi(1)}$. We have now rewritten $I^{(4)}$ as:

$$I^{(4)} = \sum_{\pi \in S(4)} \int \mathbf{d}x \exp(ip_{\pi(k)} x_k) (\theta_{12} \theta_{23} \theta_{34} + \theta_{43} \theta_{32} \theta_{21}) \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$$

where we introduced some other self-explanatory notations. To proceed, we introduce the following lemma:

Lemma. *Let $\{x_1, \dots, x_n\}$ be a set of points. We then have the following relation:*

$$\begin{aligned}
 \theta(x_1) \cdots \theta(x_n) - (-1)^n \theta(-x_1) \cdots \theta(-x_n) = \\
 [\theta(x_2) \theta(x_3) \cdots \theta(x_n)] - [\theta(-x_1) \theta(x_3) \cdots \theta(x_n)] + [\theta(-x_1) \theta(-x_2) \theta(x_4) \cdots \theta(x_n)] \\
 - [\cdots] + [(-1)^{n-1} \theta(-x_1) \theta(-x_2) \cdots \theta(-x_{n-1})]
 \end{aligned}$$

Proof. This is a direct consequence of the relation $\theta(x) = 1 - \theta(-x)$ and a straightforward induction. $\square$

It needs to be noted that this decomposition is not unique. The point of this relation is that it reduces a product of n Heaviside functions to a product to $n - 1$ Heaviside functions. In particular, for $n = 3$, we obtain:

$$\theta(x) \theta(y) \theta(z) + \theta(-x) \theta(-y) \theta(-z) = \theta(y) \theta(z) - \theta(-x) \theta(z) + \theta(-x) \theta(-y)$$

Coming back to our sheep, this implies that:

$$I^{(4)} = \sum_{\pi \in S(4)} \int \mathbf{d}x \exp(ip_{\pi(k)} x_k) (\theta_{23} \theta_{34} - \theta_{21} \theta_{34} + \theta_{21} \theta_{32}) \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$$

To illustrate what we are going to do next, let us start by an example:

$$\begin{aligned}
 & \int \mathbf{d}x \exp(ip_k x_k) \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2\} \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4\} \rangle \\
 &= \sum_{\sigma \in S(2)} \sum_{\tau \in S(2)} \int \mathbf{d}x \exp(ip_k x_k) \theta_{\sigma(1)\sigma(2)} \theta_{\tau(1)\tau(2)} \langle \mathcal{O}_{\sigma(1)} \mathcal{O}_{\sigma(2)} \mathcal{O}_{\tau(1)} \mathcal{O}_{\tau(2)} \rangle \\
 &= \sum_{\sigma \in S(2)} \sum_{\tau \in S(2)} \int \mathbf{d}x \exp(ip_{\sigma(1)} x_1 + p_{\sigma(2)} x_2 + p_{\tau(1)} x_3 + p_{\tau(2)} x_4) \theta_{12} \theta_{34} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \\
 &= \sum_{\alpha \in S^1(4)} \int \mathbf{d}x \exp(ip_{\alpha(k)} x_k) \theta_{12} \theta_{34} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle
 \end{aligned}$$

and in the same way

$$\begin{aligned}
 & \int \mathbf{d}x \exp(ip_k x_k) \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2\}^\dagger \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4\} \rangle \\
 &= \sum_{\sigma \in S(2)} \sum_{\tau \in S(2)} \int \mathbf{d}x \exp(ip_k x_k) \theta_{\sigma(1)\sigma(2)} \theta_{\tau(1)\tau(2)} \langle (\mathcal{O}_{\sigma(1)} \mathcal{O}_{\sigma(2)})^\dagger \mathcal{O}_{\tau(1)} \mathcal{O}_{\tau(2)} \rangle \\
 &= \sum_{\sigma \in S(2)} \sum_{\tau \in S(2)} \int \mathbf{d}x \exp(ip_k x_k) \theta_{\sigma(1)\sigma(2)} \theta_{\tau(1)\tau(2)} \langle \mathcal{O}_{\sigma(2)} \mathcal{O}_{\sigma(1)} \mathcal{O}_{\tau(1)} \mathcal{O}_{\tau(2)} \rangle \\
 &= \sum_{\sigma \in S(2)} \sum_{\tau \in S(2)} \int \mathbf{d}x \exp\left(i(p_{\sigma(2)} x_1 + p_{\sigma(1)} x_2 + p_{\tau(1)} x_3 + p_{\tau(2)} x_4)\right) \theta_{21} \theta_{34} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \\
 &= \sum_{\alpha \in S^1(4)} \int \mathbf{d}x \exp(ip_{\alpha(k)} x_k) \theta_{21} \theta_{34} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle
 \end{aligned}$$

We are now going to define what we mean by $S^1(4)$. We are going to decompose $S(4)$ in a very specific way. We define a fundamental permutation set

$$F = \{i^{(2)} \cdot i^{(2)}, i^{(2)} \cdot e, e \cdot i^{(2)}, e \cdot e\}$$

e is the simple transposition, i.e. it simply exchanges the elements it acts on. Be aware the dot here does not represent composition, it stands for the first term acting on the first two elements and the second acting on the third and fourth elements, e.g. $(e \cdot i^{(2)})(1234) \rightarrow 2134$.

Furthermore, let us introduce the set

$$A = \{i^{(4)}, \tau_{23}, \tau_{24}, \tau_{13}, \tau_{14}, \tau_{13} \circ \tau_{24}\}$$

where the $\tau_{ij} \in S(4)$ simply exchange elements i and j and leave the others untouched. It is then trivial to convince oneself that:

$$S(4) = F \circ A$$

This is useful because each $S^i(4) = a_i \circ F$ where $a_i \in A$ corresponds to the combination of permutations that appear in a specific operation of the T product into two T products. More specifically, we have the following where for now we focus on the second term in our previous expression for $I^{(4)}$:

$$\begin{aligned}
 - \sum_{\pi \in S(4)} \int \mathbf{d}x \exp(ip_{\pi(k)} x_k) \theta_{21} \theta_{34} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle &= - \sum_{i=1}^6 \sum_{\sigma \in S^i(4)} \int \mathbf{d}x \exp(ip_{\sigma(k)} x_k) \theta_{21} \theta_{34} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \\
 &= \int \mathbf{d}x \exp(ip_k x_k) \left(\langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2\}^\dagger \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4\} \rangle + \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_3\}^\dagger \mathbf{T}\{\mathcal{O}_2 \mathcal{O}_4\} \rangle + \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_4\}^\dagger \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_2\} \rangle \right. \\
 &\quad \left. + \langle \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_2\}^\dagger \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_4\} \rangle + \langle \mathbf{T}\{\mathcal{O}_4 \mathcal{O}_2\}^\dagger \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_1\} \rangle + \langle \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4\}^\dagger \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2\} \rangle \right)
 \end{aligned}$$

Let us now look at the examples for the two other terms:

$$\begin{aligned}
 \int \mathbf{d}x \exp(ip_k x_k) \langle \mathcal{O}_1 \mathbf{T}\{\mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle &= \sum_{\sigma \in S(3)} \int \mathbf{d}x \exp(ip_k x_k) \theta_{\sigma(1)\sigma(2)} \theta_{\sigma(2)\sigma(3)} \langle \mathcal{O}_1 \mathcal{O}_{\sigma(1)} \mathcal{O}_{\sigma(2)} \mathcal{O}_{\sigma(3)} \rangle \\
 &= \sum_{\sigma \in S(3)} \int \mathbf{d}x \exp(i(x_1 p_1 + p_{\sigma(k)} x_k)) \theta_{23} \theta_{34} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \\
 \int \mathbf{d}x \exp(ip_k x_k) \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3\}^\dagger \mathcal{O}_4 \rangle &= \sum_{\sigma \in S(3)} \int \mathbf{d}x \exp(ip_k x_k) \theta_{\sigma(1)\sigma(2)} \theta_{\sigma(2)\sigma(3)} \langle \mathcal{O}_{\sigma(3)} \mathcal{O}_{\sigma(2)} \mathcal{O}_{\sigma(1)} \mathcal{O}_4 \rangle \\
 &= \sum_{\sigma \in S(3)} \int \mathbf{d}x \exp(i(p_{\sigma(k)} x_k + p_4 x_4)) \theta_{32} \theta_{21} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle
 \end{aligned}$$

From this result, it is now trivial to conclude our final result:

$$\begin{aligned}
 I^{(4)} = \int \mathbf{d}x \exp(ip_k x_k) &\left[- \left(\langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2\}^\dagger \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4\} \rangle + \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_3\}^\dagger \mathbf{T}\{\mathcal{O}_2 \mathcal{O}_4\} \rangle + \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_4\}^\dagger \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_2\} \rangle \right. \right. \\
 &+ \langle \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_2\}^\dagger \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_4\} \rangle + \langle \mathbf{T}\{\mathcal{O}_4 \mathcal{O}_2\}^\dagger \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_1\} \rangle + \langle \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4\}^\dagger \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2\} \rangle \Big) \\
 &+ (\langle \mathcal{O}_1 \mathbf{T}\{\mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle + \langle \mathcal{O}_2 \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4 \mathcal{O}_1\} \rangle + \langle \mathcal{O}_3 \mathbf{T}\{\mathcal{O}_4 \mathcal{O}_1 \mathcal{O}_2\} \rangle + \langle \mathcal{O}_4 \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3\} \rangle) \\
 &\left. + \left(\langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3\}^\dagger \mathcal{O}_4 \rangle + \langle \mathbf{T}\{\mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\}^\dagger \mathcal{O}_1 \rangle + \langle \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4 \mathcal{O}_1\}^\dagger \mathcal{O}_2 \rangle + \langle \mathbf{T}\{\mathcal{O}_4 \mathcal{O}_1 \mathcal{O}_2\}^\dagger \mathcal{O}_3 \rangle \right) \right]
 \end{aligned}$$

To summarize, we have proven that:

$$\int \mathbf{d}x \exp(ip_k x_k) 2\text{Re}[\langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle] = \int \mathbf{d}x \exp(ip_k x_k) \sum_{\text{cuts}} \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle \quad (\text{B.1.1})$$

which finally implies:

$$2\text{Re}[\langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle] = - \sum_{\text{cuts}} \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle \quad (\text{B.1.2})$$

where by “cuts” we mean the sum represented above.

We want to find an equivalent to the optical theorem. Let us therefore now consider the regime where:

$$t = 0 \Rightarrow p_1 = -p_3 \quad \& \quad p_2 = -p_4$$

and let us look at what this implies for the Fourier transform of the above “cut”-formula:

$$\begin{aligned}
 \int \mathbf{d}x \exp(ip_k x_k) \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle^* &= \int \mathbf{d}x \exp[i(-p_3 x_1 - p_4 x_2 - p_1 x_3 - p_2 x_4)] \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle^* \\
 &= \int \mathbf{d}x \exp[-ip_k x_k] \langle \mathbf{T}\{\mathcal{O}_3 \mathcal{O}_4 \mathcal{O}_1 \mathcal{O}_2\} \rangle^* \\
 &= \left(\int \mathbf{d}x \exp[ip_k x_k] \langle \mathbf{T}\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} \rangle \right)^*
 \end{aligned}$$

where in the first line we used the fact that we are in the forward regime and in the second line we used the change of variables $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$ and the fact that operators inside a T -product commute. So that we have obtained:

$$\begin{aligned}
 \int \mathbf{d}x \exp(ip_k x_k) 2\text{Re}[\langle \mathbf{T}\{\mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4)\} \rangle] \\
 = 2\text{Re} \left[\int \mathbf{d}x \exp(ip_k x_k) \langle \mathbf{T}\{\mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4)\} \rangle \right] \quad (\text{B.1.3})
 \end{aligned}$$

The final formula of interest to us is:

$$2\text{Im} \left[\int d\mathbf{x} \exp(ip_k x_k) (-i) \langle \mathbf{T} \{ \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \} \rangle \right] = \sum_{\text{cuts}} \int d\mathbf{x} \exp(ip_k x_k) \langle \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \} \rangle \quad (\text{B.1.4})$$

Let us now look at the RHS of the equation:

$$\begin{aligned} \int d\mathbf{x} \exp(ip_k x_k) (\langle \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_2 \}^\dagger \mathbf{T} \{ \mathcal{O}_3 \mathcal{O}_4 \} \rangle) &= \sum_{|\psi\rangle} \int d\mathbf{x} \exp(ip_k x_k) \langle \Omega | \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_2 \}^\dagger | \psi \rangle \langle \psi | \mathbf{T} \{ \mathcal{O}_3 \mathcal{O}_4 \} | \Omega \rangle \\ &= \sum_{|\psi\rangle} \int d\mathbf{x} \exp(i(p_1 x_1 + p_2 x_2)) \langle \psi | \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_2 \} | \Omega \rangle^* \exp(i(p_3 x_3 + p_4 x_4)) \langle \psi | \mathbf{T} \{ \mathcal{O}_3 \mathcal{O}_4 \} | \Omega \rangle \\ &= \sum_{|\psi\rangle} \int d\mathbf{x} \exp(i(p_1 x_1 + p_2 x_2)) \langle \psi | \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_2 \} | \Omega \rangle^* \exp(-i(p_1 x_3 + p_2 x_4)) \langle \psi | \mathbf{T} \{ \mathcal{O}_3 \mathcal{O}_4 \} | \Omega \rangle \\ &= \sum_{|\psi\rangle} \left| \int d^4 x_1 d^4 x_2 \exp(-i(p_1 x_1 + p_2 x_2)) \langle \psi | \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_2 \} | \Omega \rangle \right|^2 \\ &= \sum_{|\psi\rangle} |\langle \psi | \mathbf{T} \{ \mathcal{O}(-p_1) \mathcal{O}(-p_2) \} | \Omega \rangle|^2 \end{aligned}$$

Let us look at another more problematic term:

$$\begin{aligned} \int d\mathbf{x} \exp(ip_k x_k) (\langle \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_3 \}^\dagger \mathbf{T} \{ \mathcal{O}_2 \mathcal{O}_4 \} \rangle) &= \sum_{|\psi\rangle} \int d\mathbf{x} \exp(ip_k x_k) \langle \Omega | \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_3 \}^\dagger | \psi \rangle \langle \psi | \mathbf{T} \{ \mathcal{O}_2 \mathcal{O}_4 \} | \Omega \rangle \\ &= \sum_{|\psi\rangle} \int d\mathbf{x} \exp(i(p_1 x_1 + p_3 x_3)) \langle \psi | \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_2 \} | \Omega \rangle^* \exp(i(p_2 x_2 + p_4 x_4)) \langle \psi | \mathbf{T} \{ \mathcal{O}_3 \mathcal{O}_4 \} | \Omega \rangle \\ &= \sum_{|\psi\rangle} \int d\mathbf{x} \exp(ip_1(x_1 - x_3)) \langle \psi | \mathbf{T} \{ \mathcal{O}_1 \mathcal{O}_3 \} | \Omega \rangle^* \exp(ip_2(x_2 - x_4)) \langle \psi | \mathbf{T} \{ \mathcal{O}_2 \mathcal{O}_4 \} | \Omega \rangle \\ &= \sum_{|\psi\rangle} \langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(-p_1) \} | \Omega \rangle^* \langle \psi | \mathbf{T} \{ \mathcal{O}(p_2) \mathcal{O}(-p_2) \} | \Omega \rangle \end{aligned}$$

If we now again take the one-point functions of $\mathcal{O}$ to vanish, we obtain:

$$\begin{aligned} 2\text{Im} [(-i) \langle \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(p_2) \mathcal{O}(-p_1) \mathcal{O}(-p_2) \} \rangle] &= \sum_{|\psi\rangle} \left[|\langle \psi | \mathbf{T} \{ \mathcal{O}(-p_1) \mathcal{O}(-p_2) \} | \Omega \rangle|^2 \right. \\ &\quad + \langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(-p_1) \} | \Omega \rangle^* \langle \psi | \mathbf{T} \{ \mathcal{O}(p_2) \mathcal{O}(-p_2) \} | \Omega \rangle + |\langle \psi | \mathbf{T} \{ \mathcal{O}(p_2) \mathcal{O}(-p_1) \} | \Omega \rangle|^2 \\ &\quad + |\langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(-p_2) \} | \Omega \rangle|^2 + \langle \psi | \mathbf{T} \{ \mathcal{O}(p_2) \mathcal{O}(-p_2) \} | \Omega \rangle^* \langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(-p_1) \} | \Omega \rangle \\ &\quad \left. + |\langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(p_2) \} | \Omega \rangle|^2 \right] \\ \Rightarrow 2\text{Im} [(-i) \langle \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(p_2) \mathcal{O}(-p_1) \mathcal{O}(-p_2) \} \rangle] &= \\ &\sum_{|\psi\rangle} \left(|\langle \psi | \mathbf{T} \{ \mathcal{O}(-p_1) \mathcal{O}(-p_2) \} | \Omega \rangle|^2 + |\langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(p_2) \} | \Omega \rangle|^2 \right. \\ &\quad + |\langle \psi | \mathbf{T} \{ \mathcal{O}(p_2) \mathcal{O}(-p_1) \} | \Omega \rangle|^2 + |\langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(-p_2) \} | \Omega \rangle|^2 \\ &\quad \left. + 2\text{Re} [\langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(-p_1) \} | \Omega \rangle^* \langle \psi | \mathbf{T} \{ \mathcal{O}(p_2) \mathcal{O}(-p_2) \} | \Omega \rangle] \right) \end{aligned}$$

Finally, using the fact that the ψ states have positive energy, we are finally left with:

$$2\text{Im} [(-i) \langle \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(p_2) \mathcal{O}(-p_1) \mathcal{O}(-p_2) \} \rangle] = \sum_{|\psi\rangle} |\langle \psi | \mathbf{T} \{ \mathcal{O}(p_1) \mathcal{O}(p_2) \} | \Omega \rangle|^2 \quad (\text{B.1.5})$$

B.2 Positivity of two-point functions

Let us apply exactly the same methods of the previous section to the two point function case:

$$\begin{aligned}\tilde{I}^{(2)} &\equiv \int d^4x_1 d^4x_2 \exp(ip_k x_k) 2\text{Re}[\langle \mathbf{T}\{\mathcal{O}(x_1)\mathcal{O}(x_2)\} \rangle] \\ &= \int d\mathbf{x} \exp(ip_k x_k) (\langle \mathbf{T}\{\mathcal{O}_1\mathcal{O}_2\} \rangle + \langle \mathbf{T}\{\mathcal{O}_1\mathcal{O}_2\} \rangle^*)\end{aligned}$$

The same game of redefinitions then yields:

$$\begin{aligned}\tilde{I}^{(2)} &= \sum_{\pi \in S(2)} \int d\mathbf{x} \langle \mathcal{O}_1\mathcal{O}_2 \rangle (\theta_{21} + \theta_{12}) \exp(ip_{\pi(k)} x_k) \\ &= \sum_{\pi \in S(2)} \int d\mathbf{x} \langle \mathcal{O}_1\mathcal{O}_2 \rangle \exp(ip_{\pi(k)} x_k)\end{aligned}$$

which then finally implies:

$$\text{Re}[\langle \mathbf{T}\{\mathcal{O}(x_1)\mathcal{O}(x_2)\} \rangle] = \langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle + \langle \mathcal{O}(x_2)\mathcal{O}(x_1) \rangle \quad (\text{B.2.1})$$

Notice that in the case of two point functions this could have been obtain simply by writing the definition of time-ordered product and putting together the corresponding factors.

Momentum conservation $p_1 = -p_2$ then allows to take the real part operator outside the Fourier transform, so that finally we obtain:

$$\begin{aligned}\text{Re}[\langle \mathbf{T}\{\mathcal{O}(p_1)\mathcal{O}(-p_1)\} \rangle] &= \int d\mathbf{x} e^{ip_1(x_1-x_2)} (\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle + \langle \mathcal{O}(x_2)\mathcal{O}(x_1) \rangle) \\ &= \sum_{|\psi\rangle} \int d\mathbf{x} \left(e^{ip_1x_1} \langle \Omega|\mathcal{O}(x_1)|\psi\rangle e^{-ip_1x_2} \langle \psi|\mathcal{O}(x_2)|\Omega\rangle + e^{-ip_1x_2} \langle \Omega|\mathcal{O}(x_2)|\psi\rangle e^{ip_1x_1} \langle \psi|\mathcal{O}(x_1)|\Omega\rangle \right) \\ &= \sum_{|\psi\rangle} \left(\left| \int d^4x e^{ip_1x} \langle \Omega|\mathcal{O}(x)|\psi\rangle \right|^2 + \left| \int d^4x e^{-ip_1x} \langle \Omega|\mathcal{O}(x)|\psi\rangle \right|^2 \right) \quad (\text{B.2.2})\end{aligned}$$

We have therefore also proven that:

$$\text{Im}[i\langle \mathbf{T}\{\mathcal{O}(p_1)\mathcal{O}(-p_1)\} \rangle] = \text{Re}[\langle \mathbf{T}\{\mathcal{O}(p_1)\mathcal{O}(-p_1)\} \rangle] \geq 0 \quad (\text{B.2.3})$$

which essentially corresponds to the statement that the imaginary part of two-point functions is positive.

Note the direct connection to the standard way of deriving the Källén-Lehmann representation of the two-point function. Indeed, assuming that

$$\langle \Omega|\mathcal{O}(x)|\psi\rangle = e^{-ipx} \langle \Omega|\mathcal{O}(0)|\psi\rangle \quad (\text{B.2.4})$$

holds (meaning that $\mathcal{O}$ has standard commutation relation with P), we have:

$$\begin{aligned}\langle \Omega|\mathcal{O}(x)\mathcal{O}(y)|\Omega\rangle &= \sum_{|\psi\rangle} e^{-ip_\psi(x-y)} |\langle \Omega|\mathcal{O}(0)|\psi\rangle|^2 \\ &= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \left\{ \sum_{|\psi\rangle} (2\pi)^4 \delta(p - p_\psi) |\langle \Omega|\mathcal{O}(0)|\psi\rangle|^2 \right\} \quad (\text{B.2.5})\end{aligned}$$

$$\equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} (2\pi) \theta(p^0) \rho(p^2) \quad (\text{B.2.6})$$

where we have used the fact that since the complete set of states are physical, they all satisfy $p_\psi^2 \geq 0$ and $p_\psi^o \geq 0$, which then constrains p to have the same constraints. This is the standard definition of the spectral density ρ , which is positive and real. In the above language, the above result is then rewritten as

$$\text{Im} [i\langle \mathbf{T}\{\mathcal{O}(p_1)\mathcal{O}(-p_1)\} \rangle] = \pi\rho(p_1^2) \quad (\text{B.2.7})$$

This last presentation followed [\[25\]](#).

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