



Deformations of entanglement quantities

Simon Löwe

skl89@cornell.edu

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Disclaimer

There is no claim to originality at all in this note. It should be considered as a summary, detailed accounting or collage of various papers and reviews, which we will try to refer to as much as possible.

1 Introduction

It is a firm belief of today's physicists that quantum mechanics is the basic theory needed to explain the physics of our universe. It has had an unprecedented success doing so in the past century. Nonetheless, one domain of physics has stubbornly refused unification with quantum mechanics: the most universal of all interactions, gravity. While it is true that quantum gravity bears little relevance for everyday physics, understanding it should still be considered as one of the main challenges of theoretical physics for reasons ranging from incompleteness of knowledge to the fact that it is essential for elucidating the mysteries of black holes and the Big Bang.

Einstein's general relativity is the theory which today provides our understanding of gravity as a theory of dynamical spacetime, but it does not seem to be possible to unify the latter with quantum field theories in the standard way that other unifications have taken place. It is fairly widely accepted nowadays that understanding quantum gravity will require a more radical rethinking. One of the main organizing ideas of this approach is that gravity should be "holographic"[1, 2], which roughly entails that rather than being proportional to the volume, the number of degrees of freedom in the latter are proportional to the area of the boundary of that volume. One of the most active and prolific areas of research in quantum gravity is a model which implements this notion of holography explicitly: the AdS/CFT correspondence[3, 4]. The AdS/CFT correspondence is a conjectured duality between certain type of non-gravitational quantum field theories on a fixed background spacetime in d dimensions and quantum gravity theories $(d + 1)$ dimensional asymptotically AdS spacetimes.

One of the most exciting developments of the past few years is the realization that there is a deep relationship between quantum gravity and quantum information theory. The central concept which fuelled this realization is entanglement. Entanglement is one of these quintessentially quantum mechanical concepts. It is essentially the notion that separating a quantum system into subsystems is not as trivial as we think of it in the classical world, because it turns out that subsystem can still keep information about the entire system. This apparent "non-locality" generated a lot of discussion and confusion in the early (and not so early) days of quantum mechanics and ultimately culminated in a subfield of quantum mechanics

where entanglement was seen as resource one could use to do things one couldn't do classically: quantum information.

Now interestingly enough some of the deepest questions in AdS/CFT like how does geometry emerge from QFT dynamics turned out to be intricately related to entanglement structure of states in the QFT. This was kicked off by Ryu-Takayani's (RT) proposal [5, 6] that entanglement entropy in a holographic QFT should be proportional to the area of an extremal surface extending into the AdS "bulk".

This epiphany then generated an intense and renewed interest into the study of entanglement in QFTs, both holographically and not. This is where the content of this note comes in. As one would expect, because of the non-local and global nature of entanglement entropy, computing the latter analytically is impossible for all but a few simple situations. A considerable amount of techniques and machinery goes into those computations, some of which we will present in this note. But there is a very natural question physicists have become accustomed to ask when they are stuck with complicated computations: Can I perturb the things I know by a little bit? This note is about the development of perturbation theory for entanglement computations over the past years[7, 8, 9]. The idea is to perturb the density matrix of which one is computing the entanglement by an infinitesimal amount and study the consequences. It turns out that there are two distinct interpretations to this perturbation: either one thinks of the deformation in field space, e.g. introducing an operator which makes you flow away from a UV fixed point, or one thinks of the deformation in spacetime, e.g. an actual geometric deformation of the region one is considering the entanglement of. We will mention the former but mainly focus on the latter. More precisely, the main part of the text will be a detailed account of the geometrical perturbation theory of the modular Hamiltonian, a cousin of the entanglement entropy and how it can be used to prove the Average Null Energy Condition(ANEC). ANEC is an energy condition which replaced the more standard point-wise energy conditions in general relativity, which are trivially violated in quantum mechanics. It turns out that ANEC is sufficient to prove theorems like the focussing theorems, topological censorship, etc. It is also a crucial assumption in the proof of the Hofman/Maldacena bounds.

2 Quantum entanglement

This section will provide a general introduction to entanglement and some techniques for computation. It is very strongly based on very good book by Takayangi and Rangamani[10], where you can also find the references to the original sources and proofs of the various results stated in this section. For another quick introduction, one can also look at Van Raamsdonk's lecture notes[11].

2.1 Generalities

Let us consider a quantum system with a subsystem A . We denote the complement of A by A^c . We can then decompose the Hilbert space into two tensor factors:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c} \tag{2.1}$$

Note that we could equivalently consider a bipartite system composed of systems A and B , where by definition we would have:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \quad (2.2)$$

One question one might want to ask then is given a state in the full system, what is the state of the subsystem A . In most cases the latter turns out to be a mixed state and we therefore need to work with the density matrix formalism.

If ρ is the density matrix of the full system, we define the reduced density matrix of A as:

$$\rho_A = \text{tr}_{A^c}(\rho) \quad (2.3)$$

Once one considers subsystems, one of the fascinating novelties of quantum mechanics starts playing a crucial role: quantum entanglement. We can then introduce different measures of entanglement and related quantities. The following is a non-exhaustive list:

1. The Von Neumann entropy of the reduced density matrix, called *entanglement entropy*, defined as:

$$S_A \equiv -\text{tr} \rho_A \log \rho_A \quad (2.4)$$

2. The *Renyi entropies*, which depend on a parameter α , defined as:

$$S_A^{(\alpha)} \equiv \frac{1}{1-\alpha} \log \text{tr} \rho_A^\alpha \quad (2.5)$$

3. The *modular Hamiltonian*, which was defined by analogy to the thermal case, given by:

$$\mathcal{K}_A \equiv -\log \rho_A \quad (2.6)$$

Continuing the analogy with thermodynamics, we also define modular free energy:

$$F(\rho) = \text{tr}(\rho \mathcal{K}_\rho) - S(\rho) \quad (2.7)$$

One can also define the *full modular Hamiltonian* as:

$$\hat{\mathcal{K}}_A = \mathcal{K}_A \otimes \mathbb{1}_{A^c} - \mathbb{1}_A \otimes \mathcal{K}_{A^c} \quad (2.8)$$

4. For a bipartite system, we can define the *mutual information*, motivated by subadditivity inequality:

$$I(A : B) = S_A + S_B - S_{A \cup B} \quad (2.9)$$

5. Given two density matrices ρ and σ , we define *relative entropy*:

$$S(\rho || \sigma) = \text{tr}(\rho \log \rho) - \text{tr}(\rho \log \sigma) \quad (2.10)$$

These quantities satisfy some basic general properties, which we will list here in no particular order. From here on out we use the shorthand $A_1 A_2 \equiv A_1 \cup A_2$.

- Subadditivity: For a bipartite system we have:

$$S_{A_1} + S_{A_2} \geq S_{A_1 A_2} \quad (2.11)$$

Notice that this immediately translates to:

$$I(A_1 : A_2) \geq 0 \quad (2.12)$$

- Araki-Lieb inequality: For a bipartite system we have:

$$|S_{A_1} - S_{A_2}| \leq S_{A_1 A_2} \quad (2.13)$$

Immediate corollaries are that for a pure system $S_A = S_{A^c}$ and that

$$S_{A_1 A_2} \geq \max\{S_{A_1}, S_{A_2}\} \quad (2.14)$$

- Strong subadditivity: For a tripartite system, we have:

$$S_{A_1 A_2} + S_{A_2 A_3} \geq S_{A_1 A_2 A_3} + S_{A_2} \quad (2.15)$$

$$S_{A_1 A_2} + S_{A_2 A_3} \geq S_{A_1} + S_{A_3} \quad (2.16)$$

- Renyi entropies and entanglement entropy: The relationship between Renyi entropies and the entanglement entropy is given by:

$$S_A = \lim_{\alpha \rightarrow 1} S_A^{(\alpha)} \quad (2.17)$$

- Positivity of relative entropy: As is easily inferred from the name, this states:

$$S(\rho||\sigma) \geq 0 \quad \& \quad S(\rho||\sigma) = 0 \quad \Rightarrow \quad \rho = \sigma \quad (2.18)$$

- Monotonicity of relative entropy: This states that relative entropy increases under inclusion: If $A \subset B$, then

$$S(\rho_A||\sigma_A) \leq S(\rho_B||\sigma_B) \quad \text{where} \quad \rho_\alpha = \text{tr}_{\alpha^c}(\rho), \quad \sigma_\alpha = \text{tr}_{\alpha^c}(\sigma) \quad \alpha = A, B \quad (2.19)$$

- Alternative ways of writing relative entropy: One can easily show the following:

$$S(\rho||\sigma) = F(\rho) - F(\sigma) \quad (2.20)$$

$$S(\rho||\sigma) = \Delta \langle \mathcal{K}_\sigma \rangle - \Delta \langle S \rangle \geq 0 \quad (2.21)$$

- First law of entanglement: If we perturb a reference density matrix ρ_0 (i.e. $\rho = \rho_0 + \epsilon \rho_1 + \dots$), we obtain:

$$\delta S = \delta \langle \mathcal{K}_{\rho_0} \rangle \quad (2.22)$$

2.2 Entanglement entropy in QFT

There are a lot of subtleties associated to taking the continuum limit, going from a discrete lattice system to a continuum QFT. We only mention some general facts here. For more details, refer to [10] and references therein. Our starting point is going to be very general.

We have a d -dimensional relativistic QFT on some Lorentzian, globally hyperbolic spacetime $\mathcal{B}$. We then pick a Cauchy slice Σ_{d-1} thereby defining a moment of simultaneity in the QFT. It is on this surface that the quantum state is defined, generally by a density matrix ρ_Σ . Finally, we choose a spacetime codimension-1 region A on Σ_{d-1} , which separates Σ into two parts, i.e.

$$A \cup A^c = \Sigma \quad (2.23)$$

The boundary of A is denoted ∂A and called the entangling surface. Notice that it is a codimension-2 hypersurface in $\mathcal{B}$. This is illustrated in figure 1.

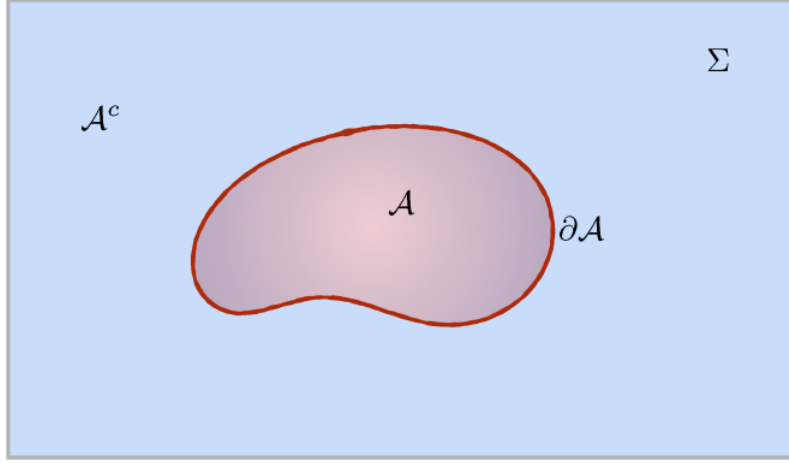


Figure 1: Illustration of various elements in the definition of entanglement in a QFT. Taken from [10].

There are now several aspects to be aware of. First of all, as it always is the case in continuum QFTs, there are going to be divergences. In this case, it is particularly apparent why: There is going to be an infinite amount of short-distance entanglement around the entangling surface. Often, this is dealt with by introducing an explicit UV regulator, which geometrically can be interpreted as "cutting out" a tubular region around the entangling surface. A second subtlety is that the notion we presented here of entanglement entropy, which measures entanglement between regions on a slice of spacetime, very much depends on locality in the sense that it needs to be possible to separate the Hilbert space into $\mathcal{H}_A \otimes \mathcal{H}_{A^c}$. This is a property which is clearly violated by gauge theories as one can easily see in a lattice discretization of the theory. It turns out that there are choices to be made and these do have consequences for entanglement entropy[12, 13, 14] e.g. boundary modes can appear[15, 16]. This issue has not been settled conclusively at this point. We will circumvent this subtlety by essentially ignoring it.

In a relativistic QFT there is one more important aspect which will play a crucial role: causality. By definition, Σ is the surface from which we can reconstruct the state in all of B . Once we are dealing with the reduced density matrix associated to A , this is no longer the case. The relevant region is then the domain of dependence of A , $D[A] = D^+[A] \cup D^-[A]$ from which the reduced density matrix of A can be uniquely evolved given the Hamiltonian on A . Once we are dealing with an entangling region, it is convenient to separate spacetime into four causally-defined regions (see figure 2):

$$\mathcal{B} = D[A] \cup D[A^c] \cup J^+[\partial A] \cup J^-[\partial A] \quad (2.24)$$

We can unitarily evolve the reduced density matrix ρ_A with operators supported

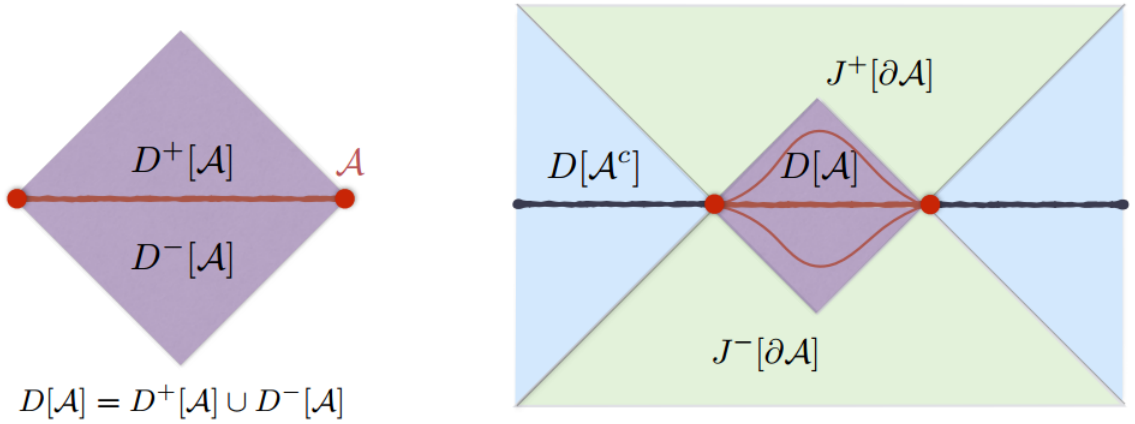


Figure 2: An illustration of the causal development and the causal decomposition of equation (2.24). Taken from [10].

on $\mathcal{H}_A$ or $\mathcal{H}_{A^c}$ without affecting the eigenvalues of ρ_A . This then implies that Renyi and von Neumann entropies are unaffected. Generally speaking the spectrum is therefore called a "wedge observable" because it only depends on $D[A]$ and not on a particular choice of Cauchy slice. One can get a lot of mileage out of these constraints on entanglement in relativistic QFTs.

2.3 The replica trick

We are now going to introduce a particular representation of states using "cut" path integrals. In this representation combined with the insight of Calabrese and Cardy [17, 18] that lead to the revival and renewed interest in entanglement. We want to define a state ρ_A on the Cauchy slice Σ . We can then at least formally represent matrix elements of the density matrix by cutting open the path integral, which means imposing boundary conditions for the fields in A :

$$\Phi_A|_{t=0^-} = \Phi_- \quad \Phi_A|_{t=0^+} = \Phi_+ \quad (2.25)$$

We then write:

$$(\rho_A)_{+-} = \int [\mathcal{D}\Phi] e^{-S[\Phi]} \delta(\Phi_A(t=0^-) - \Phi_-) \delta(\Phi_A(t=0^+) - \Phi_+) \quad (2.26)$$

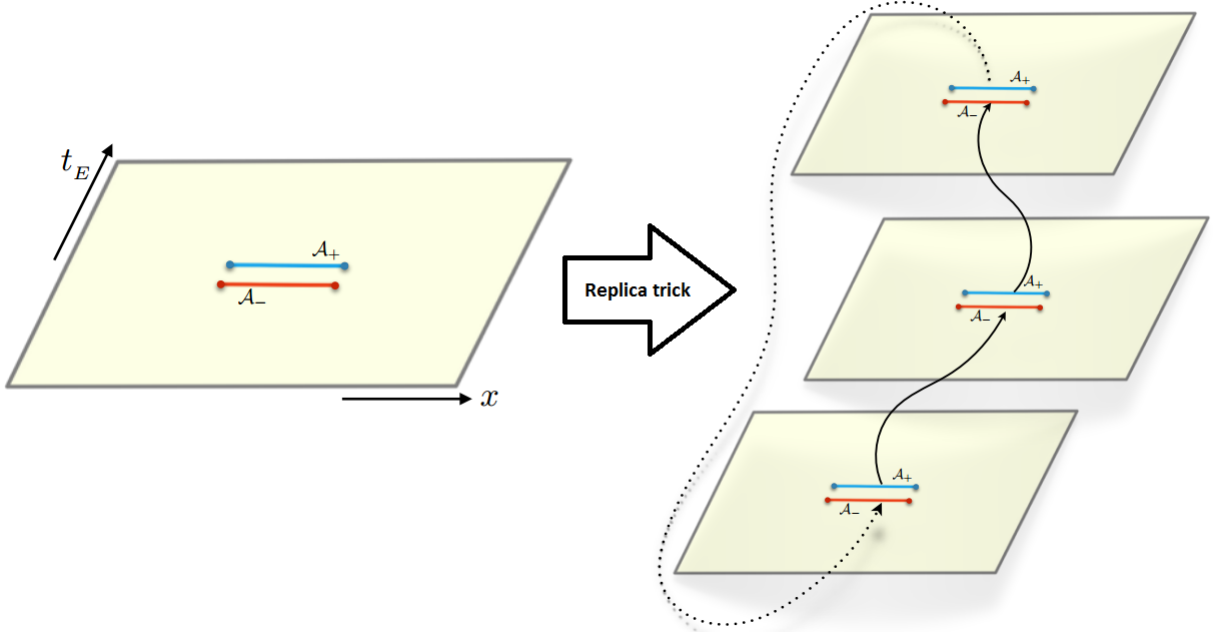


Figure 3: Illustration of "replicated" manifold. The black arrows represent the "gluing" of the various copies. Taken from [10].

It is convenient to picture this as the original spacetime with a literal cut along the region A as illustrated on the left-hand side of figure 3.

It is now straightforward to extend this representation to powers of the density matrix ρ_A^α . This is then called the replica method. Remember that matrix multiplication consists in summing a "row" of the k th density matrix with a "column" of the $(k+1)$ th. In mathematical terms this translates to:

$$(\rho_A)_{-+}^\alpha = \int \prod_{j=1}^{\alpha-1} d\Phi_+^{(j)} \delta(\Phi_+^{(j)} - \Phi_-^{(j+1)}) \quad (2.27)$$

$$\cdot \left[\int \prod_{k=1}^{\alpha} [\mathcal{D}\Phi^{(k)}] e^{-\sum S[\Phi^{(k)}]} \delta(\Phi_A^{(k)}(t=0^-) - \Phi_-) \delta(\Phi_A^{(k)}(t=0^+) - \Phi_+) \right] \quad (2.28)$$

At this point it does not seem like we have managed to simplify the situation, but a first indication of the power of this approach comes from looking at the corresponding picture. Indeed, the way to picture this construction, is by replicating the original picture α times but then connecting the slits in a crossed way, by which we mean, connecting A_+ of the k th sheet to A_- of the $(k+1)$ th sheet. Our choice of vocabulary already indicates where we are going with this. Indeed, at this point the picture should remind you of a multi-sheeted Riemann surface. The next step, which consists in taking the trace, consolidates that picture. Indeed taking the trace then glues the last two cuts. This construction is therefore analogous to the canonical construction of the α -fold branched cover over $\mathcal{B}$, which we will refer to as $\mathcal{B}_\alpha$. This construction is illustrated in figure 3.

We have therefore arrived at the following result: the quantity $\text{tr}(\rho_A^\alpha)$ can be

transformed into the partition function of the theory over a complicated multi-sheeted cover of the original space. We define

$$Z_\alpha[A] = \text{tr}(\rho_A^\alpha) \equiv Z[\mathcal{B}_\alpha] \quad (2.29)$$

This can then immediately be related to the Renyi entropies:

$$S_A^{(\alpha)} = \frac{1}{1-\alpha} \log \left(\frac{Z[\mathcal{B}_\alpha]}{Z[\mathcal{B}]^\alpha} \right) \quad (2.30)$$

3 Deformation of modular Hamiltonian

In this section, we will give an in depth and detailed accounting of geometric perturbation theory to first order of the modular Hamiltonian. This section is nothing more than more detailed version of [7].

3.1 Quick summary

Let us give a quick run-through of the argument before we dive into the gory details. One starts by noticing that a deformation can be seen as a modification of the metric, so that rather than computing the entanglement entropy of the deformed region in the old (flat) metric we compute the entanglement entropy of the old region in a new and appropriately deformed metric. The benefit of doing so is that it is well-known how to write perturbation of the metric. The latter then generates a perturbation of the density matrix in which the stress tensor plays a crucial role since the first perturbation of the action with respect to the metric is proportional to the stress-tensor.

When writing down the modular Hamiltonian, one is now faced with the problem that it entails a logarithm of the density, which is generally a very hard object to compute. To proceed, one uses the "resolvent" trick, which entails writing the logarithm as an integral of a much more tractable function of the density matrix. Through a series of clever manipulations, the resolvent parameter integral ends up becoming the real-time integral converting the Euclidean integral defining the state to an integral over the causal development of the half-space, the Rindler wedge. Finally, through an integration by parts, one is left with integrals over the boundary of the Rindler wedges, which precisely are the Rindler horizons.

The ANEC is an energy condition which takes the following form

$$\int_{-\infty}^{\infty} dx^+ \langle T_{++}(x^+, x^- = 0, \mathbf{x}) \rangle_\psi \geq 0 \quad (3.1)$$

for any state ψ . The ANEC then essentially follows immediately from the positivity of $\hat{\mathcal{K}}_{A_0} - \hat{\mathcal{K}}_A$ and the established perturbation theory result.

3.2 General setup

All of what is going to follow is about determining the modular Hamiltonian in vacuum for a region which is a slightly deformed half-space, i.e. our goal is to find an expression for $\mathcal{K}_A$.

Our natural ambient space is flat Minkowski space $\mathbb{R}^{1,d-1}$ with global coordinates $x^\mu = (x^0, x^1, \dots, x^{d-1}) \equiv (x^0, x^1, \mathbf{x})$. Following the recipe we laid out earlier, we now pick a Cauchy surface Σ specified by the equation $x^0 = 0$ and then consider a subspace of Σ , which is the half-space A_0 defined as

$$A_0 = \{x^\mu \in \mathbb{R}^{1,d-1} | x^0 = 0, x^1 > 0\} \quad (3.2)$$

It is a well-known fact[19, 20]¹ that in this case the modular Hamiltonian for the vacuum is essentially the boost generator:

$$\mathcal{K}_{A_0} = 2\pi \int_{A_0} dx^1 d^{d-2} \mathbf{x} x^1 T_{00}(0, x^1, \mathbf{x}) + A \quad (3.3)$$

Consequently, we also have:

$$\hat{\mathcal{K}}_{A_0} = 2\pi \int_{\Sigma} dx^1 d^{d-2} \mathbf{x} x^1 T_{00}(0, x^1, \mathbf{x}) \quad (3.4)$$

As stated previously, we are interested in perturbing the region A_0 geometrically. We are going to allow time-like deformations as long as A stays space-like, since it needs to be a subspace of a Cauchy surface Σ' . We parametrize that surface in the following way:

$$A = \{y^\mu \in \mathbb{R}^{1,d-1} | y^0 = \zeta^0(\mathbf{x}), y^1 = x^1 + \zeta^1(\mathbf{x})\} \subset \Sigma' \quad (3.5)$$

So ζ is the deformation vector field and we take it to point inwards, so that $\mathcal{D}(A) \subset \mathcal{D}(A_0)$ where $\mathcal{D}$ denotes the domain of dependence. These regions are illustrated in figure 4.

Following our previous explanation of how to represent states using path-integrals, the path integral representation of the state

$$|\psi\rangle = e^{-\tau H} \sum_{\alpha} c_{\alpha} \mathcal{O}_{\alpha}(0, x^1, \mathbf{x}) |0\rangle \quad (\tau > 0) \quad (3.6)$$

is a path integral over the half-space $x_E^0 < 0$ with an operator inserted at $x_E^0 = -\tau$. The corresponding reduced density matrix associated to $|\psi\rangle$ and the region A_0 is then

$$\langle \alpha_0 | \rho_{A_0, \eta}^{\psi} | \beta_0 \rangle = N_{A_0, \eta}^{-1} \int_{\phi^- = \alpha_0}^{\phi^+ = \beta_0} [\mathcal{D}\phi]_{\eta} \sum_{\alpha} \sum_{\beta} c_{\alpha}^* c_{\beta} \mathcal{O}_{\alpha}^{\dagger}(\tau, x^1, \mathbf{x}) \mathcal{O}_{\beta}(-\tau, x^1, \mathbf{x}) e^{-S[\eta, \phi(x)]} \quad (3.7)$$

where we have collectively denoted all fields of the corresponding theory with ϕ and $\phi^{\pm}$ denotes $\phi(x_E^0 \rightarrow 0^{\pm}, x^1 > 0)$. The corresponding picture is shown in figure 5.

From now on, we will use the following short-hand inside of path-integrals:

$$X = \sum_{\alpha} \sum_{\beta} c_{\alpha}^* c_{\beta} \mathcal{O}_{\alpha}^{\dagger}(\tau, x^1, \mathbf{x}) \mathcal{O}_{\beta}(-\tau, x^1, \mathbf{x}) \quad (3.8)$$

The quantity we are now interested in constructing is the reduced density matrix over the deformed region A : $\rho_{A, \eta}^{\psi}$. But thought of in this way, it is not

¹One can also prove this using path-integral arguments. For an accounting of that method, see Tom Hartman's lecture notes on AdS/CFT at <http://hartmanhep.net/>.

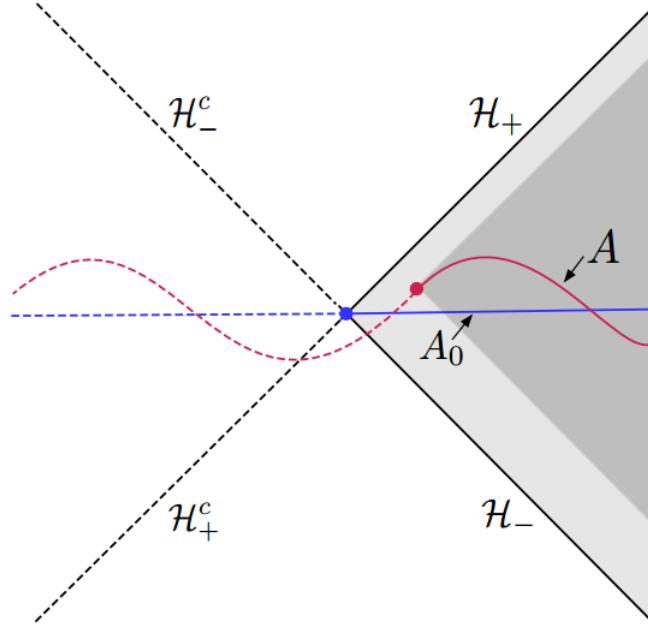


Figure 4: Illustration of the various regions. Notice that for ζ only depending on the transverse direction $\boldsymbol{x}$, the new region will only be a translation in the $x_E^0 - x^1$ plane. Taken from [7].

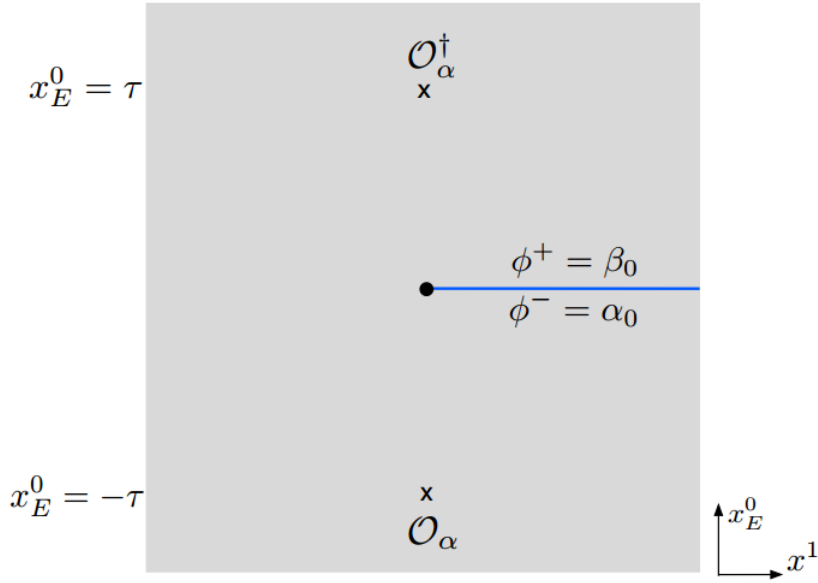


Figure 5: Pictorial representation of the density matrix for the state ψ over the half-space A_0 . The transverse direction $\boldsymbol{x}$ is implicit. Taken from [7].

clear how to devise a perturbation theory. On the other hand, what we can do is to perform a diffeomorphism

$$f : x^\mu \rightarrow x^\mu - \zeta^\mu \quad (3.9)$$

which by construction transforms A to A_0 . Now there are a few subtleties to keep in mind here. Originally, ζ is only defined on A_0 . For this to be a well-defined diffeomorphism, it needs to be defined on the entire space. But there is also another condition, that we want ζ to satisfy, which is that it is supported in Euclidean space only on a small strip around $x_E^0 = 0$, i.e. $x_E^0 < l$. These conditions though do not seem to hamper existence or generality. Of course, this diffeomorphism now transforms us away from flat Minkowski into a space with a non-trivial background metric

$$g = (f^{-1})^* \eta \quad (3.10)$$

We now claim the following result:

$$\rho_{A,\eta}^\psi = U^\dagger \rho_{A_0,g}^\psi U \quad (3.11)$$

where U is a unitary transformation. We have therefore related the object we are interested in to an object on which we know how to do perturbation theory on since we have taken ζ to be infinitesimal.

Let us now give a quick formal proof of the previous statement. We claim that the correct unitary is

$$U = \int [\mathcal{D}\alpha]_\eta |(f^{-1})^* \alpha\rangle \langle \alpha| \quad (3.12)$$

Notice that latter is unitary under the assumption of diffeomorphism invariance of the path-integral measure. The proof goes as follows:

$$\begin{aligned} \langle \alpha | \rho_{A,\eta}^\psi | \beta \rangle &= N_{A,\eta} \int_{\phi^- = \alpha}^{\phi^+ = \beta} [\mathcal{D}\phi]_\eta X e^{-S[\eta, \phi(x)]} \\ &= N_{A,\eta} \int_{(f^* \tilde{\phi})^- = \alpha}^{(f^* \tilde{\phi})^+ = \beta} [\mathcal{D}(f^* \tilde{\phi})]_\eta X e^{-S[\eta, (f^* \tilde{\phi})]} \quad \text{change of variables: } \phi = (f^* \tilde{\phi}) \\ &= N_{A_0,g} \int_{\tilde{\phi}^- = (f^{-1})^* \alpha}^{\tilde{\phi}^+ = (f^{-1})^* \beta} [\mathcal{D}\tilde{\phi}]_g X e^{-S[\eta, \tilde{\phi}]} \quad \text{diffeomorphism invariance} \\ &= \langle (f^{-1})^* \alpha | \rho_{A_0,g}^\psi | (f^{-1})^* \beta \rangle \\ &= \langle \alpha | U^\dagger \rho_{A_0,g}^\psi U | \beta \rangle \end{aligned} \quad (3.13)$$

Notice that we are again assuming that the path-integral measure is diffeomorphism invariant. Furthermore, you see in this calculation why the assumption of localized support in the Euclidean plane was important: It is so that the diffeomorphism doesn't act on the operator insertions X .

As we mentioned earlier, the entire reason for this subterfuge is to obtain a formula for which we know how to perform perturbation theory. Indeed, we have taken ζ to be infinitesimal so that we have:

$$g_{\mu\nu} = \delta_{\mu\nu} + 2\partial_{(\mu} \zeta_{\nu)} + O(\zeta^2) \equiv \delta_{\mu\nu} + \delta g_{\mu\nu} + O(\zeta^2) \quad (3.14)$$

We can now find a perturbative expression for

$$U \rho_{A,\eta}^\psi U^\dagger = \rho_{A_0,g}^\psi \quad (3.15)$$

Let us start by the expansion of the normalization factor:

$$\begin{aligned}
 N_{A_0,g} &= \int [\mathcal{D}\phi] X e^{-S[g,\phi]} \\
 &= \int [\mathcal{D}\phi] X e^{-S[g,\phi]} e^{-\int d^d x_E \delta g_{\mu\nu}(x_E) T^{\mu\nu}(x_E)} \\
 &= \left(\int [\mathcal{D}\phi] X e^{-S[\eta,\phi]} \right) \left(1 - \frac{\int d^d x_E \delta g_{\mu\nu}(x_E) \int [\mathcal{D}\phi] X T^{\mu\nu}(x_E) e^{-S[\eta,\phi]}}{\int [\mathcal{D}\phi] X e^{-S[\eta,\phi]}} + O(\zeta^2) \right)
 \end{aligned} \tag{3.16}$$

We then also have:

$$\begin{aligned}
 \int_{\phi^-=\alpha}^{\phi^+=\beta} [\mathcal{D}\phi] X e^{-S[g,\phi]} &= \int_{\phi^-=\alpha}^{\phi^+=\beta} [\mathcal{D}\phi] X e^{-S[g,\phi]} e^{-\int d^d x_E \delta g_{\mu\nu}(x_E) T^{\mu\nu}(x_E)} \\
 &= \int_{\phi^-=\alpha}^{\phi^+=\beta} [\mathcal{D}\phi] X e^{-S[g,\phi]} - \int d^d x_E \delta g_{\mu\nu}(x_E) \int_{\phi^-=\alpha}^{\phi^+=\beta} [\mathcal{D}\phi] X T^{\mu\nu}(x_E) e^{-S[g,\phi]}
 \end{aligned} \tag{3.17}$$

Now using

$$\int [\mathcal{D}\phi] X e^{-S[\eta,\phi]} = \langle X \rangle \tag{3.18}$$

and the fact that the path-integrals generate ordered correlators (in this case angular ordering), we obtain:

$$U \rho_{A,\eta}^\psi U^\dagger = \rho_{A_0,\eta}^\psi - \frac{1}{2} \int d^d x_E \delta g_{\mu\nu}(x_E) \rho_{A_0,\eta} \left(\frac{\mathcal{T}(T^{\mu\nu}(x_E) X)}{\langle X \rangle} - \frac{\langle T^{\mu\nu}(x_E) X \rangle X}{\langle X \rangle^2} \right) + O(\zeta^2) \tag{3.19}$$

If we take $|\psi\rangle = |0\rangle$, we obtain:

$$U \rho_{A,\eta} U^\dagger = \rho_{A_0,\eta} - \frac{1}{2} \int d^d x_E \delta g_{\mu\nu}(x_E) \rho_{A_0,\eta} (T^{\mu\nu}(x_E) - \langle T^{\mu\nu}(x_E) \rangle) + O(\zeta^2) \tag{3.20}$$

$$\equiv \rho_{A_0,\eta} - \frac{1}{2} \int d^d x_E \delta g_{\mu\nu}(x_E) \rho_{A_0,\eta} : T^{\mu\nu}(x_E) : + O(\zeta^2) \tag{3.21}$$

$$\equiv \rho_{A_0,\eta} - \delta \rho \tag{3.22}$$

3.3 Modular Hamiltonian

We now want to compute the modular Hamiltonian over A for the vacuum

$$\mathcal{K}_{A,\eta} \equiv -\ln \rho_{A,\eta} = -U^\dagger (\ln \rho_{A_0,g}) U = U^\dagger \mathcal{K}_{A_0,g} U \tag{3.23}$$

Computing the logarithm of an operator is an arduous task, but one way of potentially simplifying it, is by using the resolvent trick, which is an identity which says

$$\ln \rho_{A_0,g} = -\int_0^\infty d\lambda \left[(\rho_{A_0,g} + \lambda)^{-1} - \frac{1}{1+\lambda} \right] \tag{3.24}$$

Now combining this and equation (3.22), we get:

$$\begin{aligned}
 \mathcal{K}_{A_0,g} &= \int_0^\infty d\lambda \left[(\rho_{A_0,g} + \lambda)^{-1} - \frac{1}{1+\lambda} \right] \\
 &= \int_0^\infty d\lambda \left[(\rho_{A_0,\eta} + \delta\rho + \lambda)^{-1} - \frac{1}{1+\lambda} \right] \\
 &= \int_0^\infty d\lambda \left[(\mathbb{1} + (\rho_{A_0,\eta} + \lambda)^{-1} \delta\rho)^{-1} (\rho_{A_0,\eta} + \lambda)^{-1} - \frac{1}{1+\lambda} \right] \\
 &= \int_0^\infty d\lambda \left[(\mathbb{1} - (\rho_{A_0,\eta} + \lambda)^{-1} \delta\rho) (\rho_{A_0,\eta} + \lambda)^{-1} - \frac{1}{1+\lambda} \right] \\
 &= \mathcal{K}_{A_0,\eta} - \int_0^\infty d\lambda \left[(\rho_{A_0,\eta} + \lambda)^{-1} \delta\rho (\rho_{A_0,\eta} + \lambda)^{-1} \right] \\
 &= \mathcal{K}_{A_0,\eta} - \frac{1}{2} \int_0^\infty d\lambda \int d^d x_E \delta g_{\mu\nu}(x_E) \rho_{A_0,\eta} (\rho_{A_0,\eta} + \lambda)^{-1} : T^{\mu\nu}(x_E) : (\rho_{A_0,\eta} + \lambda)^{-1} \\
 &\equiv \mathcal{K}_{A_0,\eta} + \delta_\zeta \mathcal{K}_{A_0}
 \end{aligned} \tag{3.25}$$

From now on we will simply write:

$$\rho_{A_0,\eta} \equiv \rho_0 \tag{3.26}$$

To deal with the factors in the perturbation we now use the spectral representation, i.e. we introduce eigenstates of modular Hamiltonian:

$$\mathcal{K}_{A_0} |\omega_a\rangle = \omega_a |\omega_a\rangle \quad \Rightarrow \quad \rho_0 |\omega_a\rangle = e^{-\omega_a} |\omega_a\rangle \tag{3.27}$$

We then have:

$$\begin{aligned}
 -\delta_\zeta \mathcal{K}_{A_0} &= \frac{1}{2} \int_0^\infty d\lambda \int d^d x_E \int d\omega_a \int d\omega_b \delta g_{\mu\nu}(x_E) |\omega_a\rangle \langle \omega_a| \rho_{A_0,\eta} (\rho_{A_0,\eta} + \lambda)^{-1} \\
 &\quad \times : T^{\mu\nu}(x_E) : |\omega_b\rangle \langle \omega_b| (\rho_{A_0,\eta} + \lambda)^{-1} \\
 &= \frac{1}{2} \int_0^\infty d\lambda \int d^d x_E \int d\omega_a \int d\omega_b \delta g_{\mu\nu}(x_E) \frac{e^{-\omega_a}}{(e^{-\omega_a} + \lambda)(e^{-\omega_b} + \lambda)} (: T^{\mu\nu}(x_E) :)_{ab} |\omega_a\rangle \langle \omega_b| \\
 &= \frac{1}{2} \int d^d x_E \int d\omega_a \int d\omega_b \delta g_{\mu\nu}(x_E) \frac{\omega_b - \omega_a}{1 - e^{-(\omega_b - \omega_a)}} (: T^{\mu\nu}(x_E) :)_{ab} |\omega_a\rangle \langle \omega_b|
 \end{aligned} \tag{3.28}$$

We now have an kernel in ω_i s which looks like

$$\frac{\nu}{1 - e^{-\nu}} \tag{3.29}$$

where we defined $\nu = \omega_b - \omega_a$. The problem with this expression is that it isn't particularly helpful for simplifying the expression as it would still contain inverses of ρ_0 . Now it turns out that there exists the following identity:

$$\frac{\nu}{1 - e^{-\nu}} = - \int_{-\infty}^{\infty} ds \frac{e^{-i\frac{\nu}{2\pi}s}}{4 \sinh^2\left(\frac{s+i\epsilon}{2}\right)} \tag{3.30}$$

Showing this is a straightforward contour integration. Indeed, there are residues constantly spaced on the imaginary axis with residue:

$$- \frac{ie^{n\nu}\nu}{2\pi} \tag{3.31}$$

Summing these up from 1 to ∞ or 0 to $-\infty$ depending on the sign of ν , then yields the desired result.

Let us also mention some ways one could come up with this identity. One could notice that the ω kernel is the generating functional for the zeroth Bernoulli number and then use the identity which relates Bernoulli numbers and $\sinh^2$. Another way of proceeding, which generalizes easier, is to think of taking the Fourier transform.

However one proceeds, what we have gained from that formula is that now all the ω dependence is in an exponential. We can therefore write:

$$\begin{aligned}
\delta_\zeta \mathcal{K}_{A_0} &= \frac{1}{2} \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \int d^d x_E \int d\omega_a \int d\omega_b \delta g_{\mu\nu}(x_E) e^{-i \frac{\nu}{2\pi} s} (: T^{\mu\nu}(x_E) :)_{ab} |\omega_a\rangle \langle \omega_b| \\
&= \frac{1}{2} \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \int d^d x_E \\
&\quad \times \int d\omega_a \int d\omega_b \delta g_{\mu\nu}(x_E) |\omega_a\rangle \langle \omega_a| e^{-i \frac{\omega_b}{2\pi} s} : T^{\mu\nu}(x_E) : e^{-i \frac{\omega_b}{2\pi} s} |\omega_b\rangle \langle \omega_b| \\
&= \frac{1}{2} \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \int d^d x_E \\
&\quad \times \int d\omega_a \int d\omega_b \delta g_{\mu\nu}(x_E) |\omega_a\rangle \langle \omega_a| \rho_0^{-\frac{is}{2\pi}} : T^{\mu\nu}(x_E) : \rho_0^{\frac{is}{2\pi}} |\omega_b\rangle \langle \omega_b| \\
&= \frac{1}{2} \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \int d^d x_E \delta g_{\mu\nu}(x_E) \rho_0^{-\frac{is}{2\pi}} : T^{\mu\nu}(x_E) : \rho_0^{\frac{is}{2\pi}} \quad (3.32)
\end{aligned}$$

We have managed to show a very interesting result. Remember that ρ_0^{is} generates evolution in Rindler time, so we see that the stress-tensor gets evolved away from the Euclidean section into real time, but the price we have to pay is a hyperbolic kernel.

Now we want to attack the Euclidean integral, but there is a subtlety one should not forget. This expression we found originally came from the path-integral construction of the density matrix, which means that the Euclidean space we are integrating over is cut along the half-space, introducing a sort of multi-dimensional branch cut.

Continuing our computation, we now split the Euclidean space into two parts: a small solid cylinder R_b of radius b around the entangling and its complement $\tilde{R}$. It turns out that the contribution of the latter will vanish as we will show. Using the explicit expression of δg in terms of ζ , we find:

$$\delta_\zeta \mathcal{K}_0 = \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \int d^d x_E \rho_0^{-\frac{is}{2\pi}} \left(- \int_{\tilde{R}} (\partial^\mu T_{\mu\nu}) \zeta^\nu + \int_{\partial \tilde{R}} : T_{\mu\nu} : \zeta^\nu d\Sigma^\mu \right) \rho_0^{\frac{is}{2\pi}} + \delta_\zeta \mathcal{K}_b \quad (3.33)$$

where $\delta_\zeta \mathcal{K}_b$ is the contribution from inside the cylinder R_b . The divergence of stress-tensor is proportional to contact terms at the positions of the other operator insertions, but since we have chosen ζ to not have support there, that term drops out. The boundary term has two types of contributions: one from a cylinder of radius b coming from ∂R_b , we call this term the imaginary cutoff surface, and a second one from above and below A_0 , which we call the branch cut surfaces $\partial \tilde{R}_\pm$. This is illustrated in figure 6.

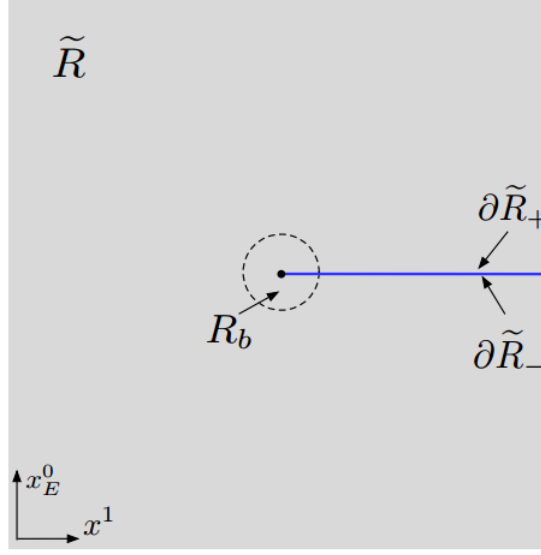


Figure 6: Split of the integration region. Taken from [7].

Let us start with the contribution from the imaginary cutoff surface. For simplicity, we switch to complex coordinates

$$z = x^1 - ix_E^0 \quad \& \quad \bar{z} = -x^1 - ix_E^0 \quad (3.34)$$

We also need the inward point unit normal to ∂R_b , which works out to be

$$n = e^{i\theta} \partial_{\bar{z}} - e^{-i\theta} \partial_z \quad (3.35)$$

We also have:

$$\zeta = \zeta^z \partial_z + \zeta^{\bar{z}} \partial_{\bar{z}} \quad (3.36)$$

We then obtain

$$\begin{aligned} \rho_0^{-\frac{is}{2\pi}} (T_{\mu\nu} n^\mu \zeta^\nu) \rho_0^{\frac{is}{2\pi}} &= \rho_0^{-\frac{is}{2\pi}} (-T_{zz}(x) \zeta^z e^{-i\theta} + T_{z\bar{z}} \zeta^z e^{i\theta} - T_{z\bar{z}} e^{-i\theta} \zeta^{\bar{z}} + T_{\bar{z}\bar{z}} e^{i\theta} \zeta^{\bar{z}}) \rho_0^{\frac{is}{2\pi}} \\ &= \left(-e^{2s-i\theta} T_{zz}(x_s) + T_{z\bar{z}}(x_s) e^{i\theta} \right) \zeta^z + \left(-T_{z\bar{z}}(x_s) e^{-i\theta} + e^{-2s+i\theta} T_{\bar{z}\bar{z}}(x_s) \right) \zeta^{\bar{z}} \end{aligned} \quad (3.37)$$

In the last line we used the fact that $\rho_0^{-is/2\pi}$ is literally the boost operator. Therefore acting on tensor indices, it generates a factor of e^s for z indices and e^{-s} for $\bar{z}$ indices since it is diagonal in $z\bar{z}$ space. Acting on the coordinates, we have:

$$z = b e^{-i\theta} \longrightarrow z_s = b e^{-i\theta+s} \longrightarrow x_s^\mu = (b \sin(\theta + is), b \cos(\theta + is), \mathbf{x}) \quad (3.38)$$

We now have the following integral:

$$\begin{aligned} \delta_\zeta \mathcal{K}_{A_0} |_{\partial R_b} &= \int d^{d-2} \mathbf{x} b \int_{-\infty}^{\infty} ds \int_0^{2\pi} d\theta \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \\ &\quad \times \left[\left(-e^{2s-i\theta} T_{zz}(x_s) + :T_{z\bar{z}}:(x_s) e^{i\theta} \right) \zeta^z + \left(-:T_{z\bar{z}}:(x_s) e^{-i\theta} + e^{-2s+i\theta} T_{\bar{z}\bar{z}}(x_s) \right) \zeta^{\bar{z}} \right] \end{aligned} \quad (3.39)$$

We can start by switching the θ and s integrals. We can do this by standard analyticity and convergence arguments. Notice that the stress-tensors depend on θ which is keeping us from doing that integral. We can solve that problem by shifting the s integration contour $s \rightarrow s + i\theta$ (this doesn't hit a pole from the $\sinh$ since $\theta < 2\pi$) which yields

$$\begin{aligned} \delta_\zeta \mathcal{K}_{A_0}|_{\partial R_b} = & -b \int d^{d-2} \mathbf{x} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\theta}{2} \right)} \\ & \times \left[\left(e^{2s} T_{zz}(\tilde{x}_s) - :T_{z\bar{z}}: (\tilde{x}_s) \right) e^{i\theta} \zeta^z + \left(:T_{z\bar{z}}: (\tilde{x}_s) - e^{-2s} T_{\bar{z}\bar{z}}(\tilde{x}_s) \right) \zeta^{\bar{z}} e^{-i\theta} \right] \end{aligned} \quad (3.40)$$

where

$$\tilde{x}_s = (ib \sinh(s), b \cosh(s), \mathbf{x}) \quad (3.41)$$

We can now perform the θ integral. The relevant integral is

$$\int_0^{2\pi} d\theta \frac{1}{4 \sinh^2 \left(\frac{s+i\theta}{2} \right)} e^{\pm i\theta} = 2\pi e^{\mp s} \Theta(\pm s) - 2\pi \delta(s) \quad (3.42)$$

Proving the first part of that integral amounts to a straightforward contour integration. The second part is trickier. It clearly amounts to some sort of principal value. So one way of proceeding is:

$$\int_0^{2\pi} \frac{1}{4 \sinh \left(\frac{s+i\theta}{2} \right)^2} e^{\pm i\theta} = \frac{d}{ds} \left[\int_0^{2\pi} \left(-\frac{1}{2} \frac{\cosh \left(\frac{s+i\theta}{2} \right)}{\sinh \left(\frac{s+i\theta}{2} \right)} \right) e^{\pm i\theta} \right] \quad (3.43)$$

The integral inside the integral can be done by taking $z = e^{i\theta}$ for $s \neq 0$ and yields a Heaviside function because the only pole at 1 is either inside or outside the contour depending on the sign of s . When $s = 0$ the pole is on the contour. Then a principal value contour deformation computes it to be some constant value. Taking the derivative yields the right result.

The delta function term reorganizes into

$$2\pi b \int d^{d-2} \mathbf{x} T_{1\mu} \zeta^\mu \quad (3.44)$$

which does not contribute in the $b \rightarrow 0$ limit. We have therefore now obtained

$$\delta_\zeta \mathcal{K}_{A_0}|_{\partial R_b} = 2\pi b \int d^{d-2} \mathbf{x} \left(- \int_0^\infty ds (e^s T_{zz} - e^{-s} :T_{z\bar{z}}:) \zeta^z + \int_{-\infty}^0 ds (e^{-s} T_{\bar{z}\bar{z}} - e^s :T_{z\bar{z}}:) \zeta^{\bar{z}} \right) \quad (3.45)$$

At first glance, this looks like it will vanish in the $b \rightarrow 0$ limit, but this turns out to not be true because of the exponential enhancement of the diagonal terms. The off-diagonal terms indeed vanish in the class of states we are considering. Let us now look at one of the diagonal terms:

$$\begin{aligned} b \int_0^\infty ds e^s T_{zz}(ib \sinh(s), b \cosh(s)) \zeta^z &= \int_b^\infty d\beta T_{zz} \left(\frac{i}{2} \beta - \frac{ib^2}{2\beta}, \frac{\beta}{2} + \frac{b^2}{2\beta} \right) \zeta^z \quad \text{where } \beta = be^s \\ &= \int_0^\infty d\beta T_{zz} \left(\frac{i}{2} \beta, \frac{1}{2} \beta \right) \zeta^z \quad b \rightarrow 0 \\ &= \int_0^\infty d\beta T_{++} \left(\frac{\beta}{2}, \frac{\beta}{2} \right) \zeta^+ \quad \text{Wick rotation} \\ &= \int_0^\infty dx^+ T_{++}(x^+, x^- = 0) \zeta^+ \end{aligned} \quad (3.46)$$

Putting everything together, we have obtained

$$\delta_\zeta \mathcal{K}_{A_0}|_{\partial R_b} = 2\pi b \int d^{d-2} \mathbf{x} \left(- \int_0^\infty dx^+ T_{++}(x^+, x^- = 0, \mathbf{x}) \zeta^+ + \int_{-\infty}^0 dx^- T_{--}(x^+ = 0, x^-, \mathbf{x}) \zeta^- \right) \quad (3.47)$$

As we already mentioned earlier, the s integral which came from the resolvent trick morphed into a real-time integral transforming the Euclidean integral which was defining the state into a real-time integral. This does suggest that there should be a way of computing this without all these bells and whistles.

We now move on to the "branch cut" term. We have:

$$\begin{aligned} & \lim_{\epsilon_+ \rightarrow 0} \lim_{\epsilon_- \rightarrow 0} \int d^{d-2} \mathbf{x} \int_b^\infty dx^1 \int_{-\infty}^\infty ds \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \left(t_\uparrow^\mu \zeta^\nu \rho_0^{-is/2\pi} : T_{\mu\nu}(x^1 \sin \epsilon_+, x^1 \cos \epsilon_+, \mathbf{x}) : \rho_0^{is/2\pi} \right. \\ & \quad \left. + t_\downarrow^\mu \zeta^\nu \rho_0^{-is/2\pi} : T_{\mu\nu}(x^1 \sin(-\epsilon_-), x^1 \cos(-\epsilon_-), \mathbf{x}) : \rho_0^{is/2\pi} \right) \\ &= \lim_{\epsilon_+ \rightarrow 0} \lim_{\epsilon_- \rightarrow 0} \int d^{d-2} \mathbf{x} \int_b^\infty dx^1 \int_{-\infty}^\infty ds \left(\frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} t_\uparrow^\mu \zeta^\nu \rho_0^{-is/2\pi} : T_{\mu\nu}(x^1 \sin \epsilon_+, x^1 \cos \epsilon_+, \mathbf{x}) : \rho_0^{is/2\pi} \right. \\ & \quad \left. + \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} t_\downarrow^\mu \zeta^\nu \right. \\ & \quad \left. \times \rho_0^{-i(s+i(\epsilon_++\epsilon_-)-2\pi i)/2\pi} : T_{\mu\nu}(x^1 \sin \epsilon_+, x^1 \cos \epsilon_+, \mathbf{x}) : \rho_0^{i(s+i(\epsilon_++\epsilon_-)-2\pi i)/2\pi} \right) \\ &= \lim_{\epsilon_+ \rightarrow 0} \lim_{\epsilon_- \rightarrow 0} \int d^{d-2} \mathbf{x} \int_b^\infty dx^1 \int_{-\infty}^\infty ds \left(\frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} t_\uparrow^\mu \zeta^\nu \rho_0^{-is/2\pi} : T_{\mu\nu}(x^1 \sin \epsilon_+, x^1 \cos \epsilon_+, \mathbf{x}) : \rho_0^{is/2\pi} \right. \\ & \quad \left. + \frac{1}{4 \sinh^2 \left(\frac{s+i(\epsilon_+-\epsilon_-)}{2} \right)} t_\downarrow^\mu \zeta^\nu \rho_0^{-is/2\pi} : T_{\mu\nu}(x^1 \sin \epsilon_+, x^1 \cos \epsilon_+, \mathbf{x}) : \rho_0^{is/2\pi} \right) \\ &= \lim_{\epsilon_+ \rightarrow 0} \lim_{\epsilon_- \rightarrow 0} \int d^{d-2} \mathbf{x} \int_b^\infty dx^1 \int_{-\infty}^\infty ds \left(\frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} - \frac{1}{4 \sinh^2 \left(\frac{s+i(\epsilon_+-\epsilon_-)}{2} \right)} \right) \\ & \quad \times t_\uparrow^\mu \zeta^\nu \rho_0^{-is/2\pi} : T_{\mu\nu}(0^+, x^1, \mathbf{x}) : \rho_0^{is/2\pi} \\ &= \lim_{\epsilon \rightarrow 0} \int d^{d-2} \mathbf{x} \int_b^\infty dx^1 \int_{-\infty}^\infty ds \left(\frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} - \frac{1}{4 \sinh^2 \left(\frac{s-i\epsilon}{2} \right)} \right) \\ & \quad \times t_\uparrow^\mu \zeta^\nu \rho_0^{-is/2\pi} : T_{\mu\nu}(0^+, x^1, \mathbf{x}) : \rho_0^{is/2\pi} \end{aligned}$$

The first line is simply writing out the term which has two contributions from above and below the cut. In the second line we used the fact that ρ_0^{is} is the rotation operator in the Euclidean plane and therefore by rotating by $\epsilon_+ + \epsilon_- - 2\pi$ we bring the stress-tensor who was previously evaluated below the cut, above the cut. In the third line, we shift the contour to move the ϵ s into the kernel. From the last expression it is clear that we simply pick out the double pole at $s = 0$. Using the formula:

$$\partial_s (e^{As} B e^{-As})|_{s=0} = [A, B] \quad (3.48)$$

we finally obtain

$$\delta_\zeta \mathcal{K}_{A_0}|_{\partial \tilde{R}_\pm} = 2\pi \int d^{d-2} \mathbf{x} \int_b^\infty dx^1 t^\mu \zeta^\nu \left[\frac{\mathcal{K}_{A_0}}{2\pi}, T_{\mu\nu}(0^+, x^1, \mathbf{x}) \right] \quad (3.49)$$

Finally, we have to deal with term from inside the imaginary cutoff surface. This term looks like follows

$$\int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\epsilon}{2} \right)} \int_{R_b} d^d x_E \partial^\mu \zeta^\nu \rho_0^{-\frac{is}{2\pi}} : T_{\mu\nu}(x_E) : \rho_0^{\frac{is}{2\pi}} \quad (3.50)$$

We now go through the same process as for the imaginary cutoff term. We end with

$$\rho_0^{-\frac{is}{2\pi}} : T_{\mu\nu}(x_E) : \rho_0^{\frac{is}{2\pi}} = e^{i\theta+s} T_{iz}(x_s) \partial_i \zeta^z + e^{-i\theta-s} T_{i\bar{z}}(x_s) \partial_i \zeta^{\bar{z}} \quad (3.51)$$

where

$$x_s^\mu = (r \sin(\theta + is), r \cos(\theta + is), \mathbf{x}) \quad (3.52)$$

If we now do the Wick rotation, we end up with

$$\rho_0^{-\frac{is}{2\pi}} : T_{\mu\nu}(x_E) : \rho_0^{\frac{is}{2\pi}} = e^{i\theta+s} T_{i+}(x^+, x^-, \mathbf{x}) \partial_i \zeta^+ + e^{-i\theta-s} T_{i-}(x^+, x^-, \mathbf{x}) \partial_i \zeta^- \quad (3.53)$$

where

$$x^\pm = \pm r e^{\pm s} \quad (3.54)$$

The full expression now is

$$\begin{aligned} \int d^{d-2} \mathbf{x} \int_{r < b} dr r \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s+i\theta}{2} \right)} \\ \times \left(e^{i\theta+s} T_{i+}(x^+, x^-, \mathbf{x}) \partial_i \zeta^+ + e^{-i\theta-s} T_{i-}(x^+, x^-, \mathbf{x}) \partial_i \zeta^- \right) \end{aligned} \quad (3.55)$$

As before we can perform the θ integral and obtain

$$\int d^{d-2} \mathbf{x} \int_{r < b} dr r \left[\int_0^\infty ds T_{i+}(x^+, x^-, \mathbf{x}) \partial_i \zeta^+ + \int_{-\infty}^0 ds T_{i-}(x^+, x^-, \mathbf{x}) \partial_i \zeta^- \right] \quad (3.56)$$

We dropped the delta function contribution because as before it vanishes in the $b \rightarrow 0$ limit. The s integrals now become literally the real-time x^0 so that we can finally write

$$\delta_\zeta \mathcal{K}_b = 2\pi \int_{0 < r < b} d^d x T_{\mu\nu} \partial^\mu \tilde{\zeta}^\nu \quad (3.57)$$

where the integral is over the neighbourhood of the Rindler horizon depicted in figure 7 and the vector field is defined as

$$\tilde{\zeta} = \Theta(x^0) \zeta^+ \partial_+ + \Theta(-x^0) \zeta^- \partial_- \quad (3.58)$$

That integral is clearly zero, since it has no enhancements. One could also do an integration by part and see that the boundary term on the imaginary surface cancels the one we computed previously and one is only left with Rindler horizon boundary terms. Putting everything together, we have shown

$$U \mathcal{K}_A U^\dagger = \mathcal{K}_{A_0} - 2\pi \int_{H_+} \zeta^+ T_{++} + 2\pi \int_{H_-} \zeta^- T_{--} + 2\pi \int_{A_0} t^\mu \zeta^\nu [\mathcal{K}_{A_0}, T_{\mu\nu}] \quad (3.59)$$

and in exactly the same way one obtains the formula for the complement

$$V \mathcal{K}_{A^c} V^\dagger = \mathcal{K}_{A_0^c} + 2\pi \int_{H_+^c} \zeta^+ T_{++} - 2\pi \int_{H_-^c} \zeta^- T_{--} + 2\pi \int_{A_0^c} t^\mu \zeta^\nu [\mathcal{K}_{A_0^c}, T_{\mu\nu}] \quad (3.60)$$

These two expressions can be combined to obtain the expression for the full modular Hamiltonian

$$\mathcal{U} \hat{\mathcal{K}}_A \mathcal{U}^\dagger = \hat{\mathcal{K}}_{A_0} - 2\pi \int_{L_+} \zeta^+ T_{++} + 2\pi \int_{L_-} \zeta^- T_{--} + 2\pi \int_{\Sigma} t^\mu \zeta^\nu [\hat{\mathcal{K}}_{A_0}, T_{\mu\nu}] \quad (3.61)$$

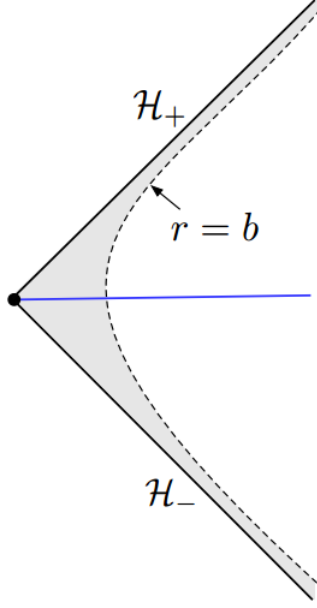


Figure 7: The region inside the imaginary cutoff surface after the transformations. Taken from [7].

3.4 Average Null Energy Condition

We start by showing the "positivity" of the operator $\hat{\mathcal{K}}_{A_0} - \hat{\mathcal{K}}_A$. The quotations are there because the two operators act on a priori different Hilbert spaces, so one has to be a little careful about what is meant by that statement.

Let us consider two states: the vacuum and a non-trivial pure state $|\psi\rangle$. From formula (2.21), we have

$$S(\rho_{A_0}^\psi \| \rho_{A_0}) = \Delta \langle \mathcal{K}_{A_0} \rangle - \Delta S_{A_0} \quad (3.62)$$

where as before $\mathcal{K}_{A_0}$ is the vacuum modular Hamiltonian. Among the many interesting properties of relative entropy, the one which we need for our purposes is monotonicity: Take $A \subset A_0$, then we have

$$S(\rho_A^\psi \| \rho_A) \leq S(\rho_{A_0}^\psi \| \rho_{A_0}) \quad (3.63)$$

If now write out this formula for both A and its complement A^c using (2.21), we obtain:

$$\Delta \langle \mathcal{K}_A \rangle - \Delta \langle \mathcal{K}_{A_0} \rangle - \Delta S_A + \Delta S_{A_0} \leq 0 \quad (3.64)$$

$$\Delta \langle \mathcal{K}_{A_0^c} \rangle - \Delta \langle \mathcal{K}_{A^c} \rangle - \Delta S_{A_0^c} + \Delta S_{A^c} \leq 0 \quad (3.65)$$

Adding these two up yields

$$\Delta \langle \hat{\mathcal{K}}_A \rangle - \Delta \langle \hat{\mathcal{K}}_{A_0} \rangle - \Delta S_A + \Delta S_{A_0} - \Delta S_{A_0^c} + \Delta S_{A^c} \leq 0 \quad (3.66)$$

The vacuum contributions vanish, which follows from the fact that $\hat{\mathcal{K}}_A$ annihilates the vacuum for any region A since:

$$\langle \hat{\mathcal{K}}_A \rangle_{|0\rangle} = \text{tr}(\rho_A \mathcal{K}_A) - \text{tr}(\rho_A \mathcal{K}_{A^c}) = S_{A^c} - S_A = 0 \quad (3.67)$$

since the vacuum is a pure state. The entanglement entropies also all cancel because of the purity of both the vacuum and $|\psi\rangle$, so that we finally have obtained:

$$\langle \hat{\mathcal{K}}_A \rangle_\psi - \langle \hat{\mathcal{K}}_{A_0} \rangle_\psi \leq 0 \quad (3.68)$$

This is what we mean by "positivity" in this context.

Let us now actually compute the above quantity.

$$\begin{aligned} \langle \hat{\mathcal{K}}_A \rangle_\psi - \langle \hat{\mathcal{K}}_{A_0} \rangle_\psi &= \text{tr}_{A_0}(\rho_{A_0}^\psi \mathcal{K}_{A_0}) - \text{tr}_{A_0^c}(\rho_{A_0^c}^\psi \mathcal{K}_{A_0^c}) - \text{tr}_A(\rho_A^\psi \mathcal{K}_A) + \text{tr}_{A^c}(\rho_{A^c}^\psi \mathcal{K}_{A^c}) \\ &= \text{tr}_{A_0}(\rho_{A_0}^\psi \mathcal{K}_{A_0}) - \text{tr}_{A_0^c}(\rho_{A_0^c}^\psi \mathcal{K}_{A_0^c}) - \text{tr}_A(U^\dagger U \rho_A^\psi U^\dagger U \mathcal{K}_A U^\dagger U) + \text{tr}_{A^c}(V^\dagger V \rho_{A^c}^\psi V^\dagger V \mathcal{K}_{A^c} V^\dagger V) \\ &= \text{tr}_{A_0}(\rho_{A_0}^\psi \mathcal{K}_{A_0}) - \text{tr}_{A_0^c}(\rho_{A_0^c}^\psi \mathcal{K}_{A_0^c}) - \text{tr}_{A_0}(U \rho_A^\psi U^\dagger U \mathcal{K}_A U^\dagger) + \text{tr}_{A_0^c}(V \rho_{A^c}^\psi V^\dagger V \mathcal{K}_{A^c} V^\dagger) \\ &= \text{tr}_{A_0}(\rho_{A_0}^\psi \mathcal{K}_{A_0}) - \text{tr}_{A_0^c}(\rho_{A_0^c}^\psi \mathcal{K}_{A_0^c}) - \text{tr}_{A_0}((\rho_{A_0}^\psi + \delta_\zeta \rho_{A_0}^\psi)(\mathcal{K}_{A_0} + \delta_\zeta \mathcal{K}_{A_0})) \\ &\quad + \text{tr}_{A_0^c}((\rho_{A_0^c}^\psi + \delta_\zeta \rho_{A_0^c}^\psi)(\mathcal{K}_{A_0^c} + \delta_\zeta \mathcal{K}_{A_0^c})) \\ &= -\text{tr}_{A_0}(\delta_\zeta \rho_{A_0}^\psi \mathcal{K}_{A_0}) + \text{tr}_{A_0^c}(\delta_\zeta \rho_{A_0^c}^\psi \mathcal{K}_{A_0^c}) - \text{tr}_{A_0}(\rho_{A_0}^\psi \delta_\zeta \mathcal{K}_{A_0}) + \text{tr}_{A_0^c}(\rho_{A_0^c}^\psi \delta_\zeta \mathcal{K}_{A_0^c}) \\ &\equiv I_{\delta\rho} + I_{\delta\mathcal{K}} \end{aligned} \quad (3.69)$$

As a direct consequence of (3.59) and (3.60) we obtain

$$I_{\delta\mathcal{K}} = 2\pi \int_{L_+} \zeta^+ \langle T_{++} \rangle_\psi - 2\pi \int_{L_-} \zeta^- \langle T_{--} \rangle_\psi - 2\pi \int_\Sigma t^\mu \zeta^\nu \langle [\hat{\mathcal{K}}_{A_0}, T_{\mu\nu}] \rangle_\psi \quad (3.70)$$

Using the equation 3.22, we can for example write the first term of $I_{\delta\rho}$ as

$$\text{tr}_{A_0}(\rho_{A_0}^\psi \delta_\zeta \mathcal{K}_{A_0}) = \frac{1}{2} \int d^d x_E \delta g_{\mu\nu}(x_E) \left(\frac{\langle \mathcal{T}(T^{\mu\nu}(x_E) \mathcal{K}_{A_0} X) \rangle}{\langle X \rangle} - \frac{\langle T^{\mu\nu}(x_E) X \rangle \langle \mathcal{K}_{A_0} X \rangle}{\langle X \rangle^2} \right) \quad (3.71)$$

An integration by parts, then leaves a divergence of the stress tensor term which vanishes as before and a boundary term, which corresponds to the branch-cuts. As it does in standard QFT, this constellation of an ordered correlation function evaluated above and below a cut then simply yields

$$\text{tr}_{A_0}(\rho_{A_0}^\psi \delta_\zeta \mathcal{K}_{A_0}) = 2\pi \int_\Sigma d^{d-2} \mathbf{x} \int_0^\infty dx^1 \zeta_\mu t_\nu \langle [\mathcal{K}_{A_0}, T^{\mu\nu}(0, x^1, \mathbf{x})] \rangle_\psi \quad (3.72)$$

For the full modular Hamiltonian, we have therefore shown that it takes the following form:

$$I_{\delta\rho} = 2\pi \int_\Sigma t^\mu \zeta^\nu \langle [\hat{\mathcal{K}}_{A_0}, T_{\mu\nu}] \rangle_\psi \quad (3.73)$$

which as promised cancels the unwanted term in $I_{\delta\rho}$. We have therefore shown that

$$\langle \hat{\mathcal{K}}_A \rangle_\psi - \langle \hat{\mathcal{K}}_{A_0} \rangle_\psi = 2\pi \int_{L_+} \zeta^+ \langle T_{++} \rangle_\psi - 2\pi \int_{L_-} \zeta^- \langle T_{--} \rangle_\psi \geq 0 \quad (3.74)$$

Using that by construction we have $\zeta^+ > 0$ and $\zeta^- < 0$ and setting one of the two to zero leads to the ANECs

$$\int_{-\infty}^\infty dx^+ \langle T_{++}(x^+, x^- = 0, \mathbf{x}) \rangle_\psi \geq 0 \quad (3.75)$$

$$\int_{-\infty}^\infty dx^- \langle T_{++}(x^+ = 0, x^-, \mathbf{x}) \rangle_\psi \geq 0 \quad (3.76)$$

3.5 Comments

In this section, we make a few comments which correspond to questions and confusions we could not resolve or giving a satisfying answer to.

- At various places during the computation it was necessary to have ζ not depend on θ , r , x^+ and x^- , nonetheless ζ had to satisfy a number of conditions both in the real and Euclidean planes, making the existence of smooth ζ not entirely trivial. It is worth pointing out that it also not clear whether we actually need ζ to be smooth. One other potential way of solving this problem, which seems to be what [7] are implying, is that since we are working at leading order we can take ζ to be independent of r and the other variables at leading order.
- A crucial step in translating the problem from a geometric deformation to a metric deformation is the ability to unitarily map between the corresponding Hilbert spaces. While this does not seem like a big deal in an actual continuum theory, it is somewhat strange for the discretized theories, which after all is what we ultimately work with in QFT. Indeed, think of deforming a region on a lattice QFT: That process necessarily involves adding or subtracting degrees of freedom, and it is impossible to map unitarily between Hilbert spaces of different finite dimension. In essence, this goes back to the question of how to make sense of diffeomorphism invariance in a discretized space.
- It is also not entirely clear how necessary the entire imaginary cut off surface charade is. It seems like barring the one contact term from when the derivatives in equation 3.57 hit the Heaviside function which certainly could be dealt with, you could get exactly the same result in a more natural way.

4 Deformations of entanglement entropy

In this section, we will describe two other papers where perturbation theory on entanglement entropy was explored. The section on operator deformation is a summary of [9] and the section on shape deformation is a summary of [8].

4.1 Operator deformation

It is possible to develop a perturbation theory for entanglement entropy away from the conformal limit. More precisely, what can be done is developing a conformal perturbation theory in some relevant coupling about a UV CFT fixed point. Entanglement entropy has turned out to be a very useful tool to study renormalization group (RG) flows, and in particular c -theorems, which have been proven using entanglement in 2, 3 and 4 dimensions. In 3D for instance, there exists a monotonically decreasing function of R defined in term of the entanglement entropy of a ball of radius R

$$F(R) = RS'_B(R) - S_B(R) \quad (4.1)$$

which can be written in the small R limit as

$$F(R) = F_{UV} - \frac{2\pi^2(\Delta - 3)}{(2\Delta - 7)} \lambda^2 R^{2(3-\Delta)} + \dots \quad (4.2)$$

where Δ is the dimension of the deforming operator and λ is the corresponding coupling. One surprising fact is that this entanglement entropy in deformed CFTs can be calculated using a classical general relativity problem, which turns out to be exactly the problem you'd expect from the holographic entanglement entropy results, namely

1. Solve to first order in λ and second order in the metric perturbation the classical GR problem in one higher dimension

$$\nabla^2 \phi = \Delta(\Delta - d)\phi \quad R_{ab} - \frac{1}{2} R g_{ab} - \frac{1}{2} d(d-1) g_{ab} = 8\pi G_N T_{ab}^\phi \quad \phi \xrightarrow{z \rightarrow 0} \lambda(x) z^{d-\Delta} + \dots \quad (4.3)$$

with regular boundary conditions in the interior of the emergent space and with the metric being asymptotically AdS in Poincare coordinates with radial coordinate z as $z \rightarrow 0$.

2. The entanglement entropy is proportional to the area of a minimal surface ending on the ball shaped region at $z = 0$.

This is in some sense not to surprising since at this order, the perturbation only entails 2- and 3-point functions in the CFT, which are universal quantities and therefore make it plausible for the holographic results to be reproduced.

Let us give a quick outline of the computation. For a CFT on Euclidean space $\mathbb{R}^d$ there exists a conformal map which takes the latter to $\mathbb{S}^1 \times \mathbb{H}_{d-1}$ [21] where $\mathbb{H}_{d-1}$ is hyperbolic space. The entanglement entropy then simply becomes thermal entropy for a CFT living on spatial slices $\mathbb{H}_{d-1}$. The steps of this computation then entail integrals of 2- and 3-point functions of primaries living on $\mathbb{S}^1 \times \mathbb{H}_{d-1}$, which are fixed by conformal symmetry up to a finite set of parameters.

One starts with the replica trick which tells us to compute the Z_n partition function in the presence of the deformation as follows

$$Z_n = \int_{\mathcal{B}_n} [\mathcal{D}\phi] e^{-S[\phi] - \int \lambda \mathcal{O}} = Z_n(\lambda = 0) \left(1 + \frac{1}{2} \lambda^2 \int_a \int_b \langle \mathcal{O}(a) \mathcal{O}(b) \rangle_n + \dots \right) \quad (4.4)$$

After mapping to $\mathbb{S}^1 \times \mathcal{H}_{d-1}$, it becomes about computing a thermal Green's function on hyperbolic space with inverse temperature $2\pi n$ (since we are on the replicated manifold). There are a lot of complications associated to the analytic continuations in n . One can with quite some work show that there are two contributions to the entanglement entropy:

$$\delta S_B^{(1)} = 2\pi \int d\mu \text{tr} \left[e^{-2\pi H} H \mathcal{T} (\mathcal{O}_a(i\tau_a) \mathcal{O}_b(i\tau_b)) \right] \quad (4.5)$$

and

$$\delta S_B^{(2)} = \int d\mu \int_{-\infty}^{\infty} ds \frac{1}{4 \sinh^2 \left(\frac{s - i\tau_{ba}}{2} \right)} \text{tr} \left[e^{-2\pi H} H \mathcal{T} (\mathcal{O}_a(0) \mathcal{O}_b(s - i\epsilon \text{sgn}(\tau_{ba}))) \right] \quad (4.6)$$

where $d\mu$ is the integration measure over the two hyperbolic spaces (including the conformal factors). The idea here is that through some clever manipulations you lift the computation from the Euclidean plane into real-time and end up with 2- and 3-points functions. This is best summarized in figure 8. It turns out the first term is divergent but is cancelled by a part of the second term. At this point, it is

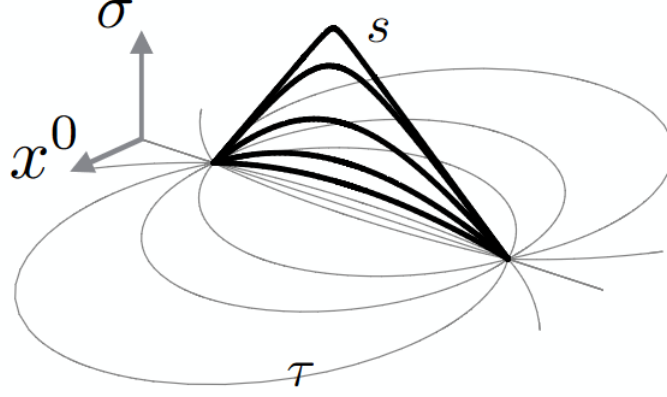


Figure 8: "The calculation of EE forces us into real times, where we integrate over the position of one of the operators after a real time modular flow transformation by an amount s . This modular flow pushes the operator into the causal development of A pictured here (thick lines are lines of constant s). This is contrasted with the modular flow by an amount τ in Euclidean time (lighter lines are lines of constant τ)."[9]

then all about doing that integral and after a terrible ordeal one ends up showing that at least one way of doing that integral is by essentially all the ingredients one would expect from holography with a final formula

$$\delta S_B = -\lambda^2 R^{2(d-\Delta)} \frac{\pi^{\frac{d+1}{2}} (d-\Delta) \Gamma(1 + \frac{1}{2}d - \Delta)}{2\Gamma(\frac{3}{2} + d - \Delta)} \quad (4.7)$$

4.2 Shape deformation

By changing the formalism presented in the previous section on deformations of the modular Hamiltonian, one can also develop a perturbation theory in geometrical deformations, i.e. study the shape dependence of entanglement entropy.

Starting from the half-space

$$A = \{x^\mu \in \mathbb{R}^{1,d-1} | x^0 = 0, x^1 > 0\} \quad (4.8)$$

one deforms that region slightly to

$$A + \delta A = \{x^\mu \in \mathbb{R}^{1,d-1} | x^0 = 0, x^1 > -\epsilon \chi(x^2, \dots, x^{d-1})\} \quad (4.9)$$

where χ is the deformation which only depends on the transverse coordinates. This is illustrated in figure 9. One can actually be a little bit more general by taking the deformation to be $\zeta^{\bar{i}}(x^i)$ where $\bar{i} = 0, 1$ to include time-like deformations. One can

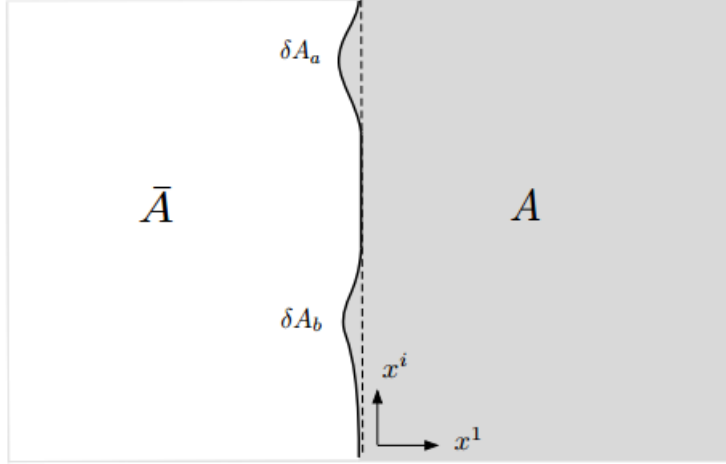


Figure 9: Schematic illustration of the deformation. Taken from [8].

then write a perturbative series for the entanglement entropy across the deformed surface

$$S[A+\delta A] = S[A] + \int d^{d-2}x \zeta^{\bar{i}}(x) S_{\bar{i}}^{(1)}(x) + \frac{1}{2!} \int d^{d-2}x_a d^{d-2}x_b \zeta^{\bar{i}}(x_a) \zeta^{\bar{j}}(x_b) S_{\bar{i}\bar{j}}^{(2)}(x_a, x_b) + \dots \quad (4.10)$$

where the quantity $S_{\bar{i}\bar{j}}^{(2)}(x_a, x_b)$ is known as the entanglement density.

In CFTs, the latter will generally contain two types of terms: contact terms in coincident $x_a \rightarrow x_b$ limit and a well-defined and finite term when x_a and x_b are separated. Indeed, one shows that for any CFT, the non-local term in the entanglement density for a planar surface is universal and given by

$$S_{\bar{i}\bar{j}, \text{non-local}}^{(2)}(x_a, x_b) = -\frac{2\pi^2 C_T}{(d+1)} \frac{\eta_{\bar{i}\bar{j}}}{|x_a - x_b|^{2(d-1)}} \quad (4.11)$$

where C_T is the numerical coefficient appearing in the 2-point function of stress tensors in CFTs i.e. the only part not fixed by conformal symmetry. This can be transformed to a formula about a ball-shaped region by the standard mapping. There are also a few direct consequences of that formula such as the corner contributions to entanglement entropy and the Mezei formula for the universal part of the entanglement entropy across deformed spheres.

The idea follows closely what has been done for the modular Hamiltonian but one has to keep one higher order. Indeed, one proves that

$$\delta S = \delta S^{(1)} + \delta S^{(2)} \quad (4.12)$$

with

$$\delta S^{(1)} = \frac{1}{2} \int d^d x \delta g_{\mu\nu}(x) \langle T^{\mu\nu}(x) \mathcal{K} \rangle_c + \frac{1}{8} \int d^d x_a d^d x_b \delta g_{\mu\nu}(x_a) \delta g_{\alpha\beta}(x_b) \langle T^{\mu\nu}(x_a) T^{\alpha\beta}(x_b) \mathcal{K} \rangle_c \quad (4.13)$$

and

$$\begin{aligned} \delta S^{(2)} = & \frac{1}{32} \int d^d x_a d^d x_b \delta g_{\mu\nu}(x_a) \delta g_{\alpha\beta}(x_b) \\ & \times \int_{-\infty}^{\infty} \frac{ds}{\sinh^2(s/2 + i\epsilon \text{sgn}(\theta_a - \theta_b))} \text{tr}_c \left(\rho_0 \mathcal{T} \left[T^{\mu\nu}(x_a) \rho_0^{-is/2\pi} T^{\alpha\beta}(x_b) \rho_0^{is/2\pi} \right] \right) \end{aligned} \quad (4.14)$$

The steps are very similar to the case of the modular Hamiltonian even though the time ordering in this case requires some more careful analytic continuations.

Notice that $\delta S^{(1)}$ is an integral of a 3-point function of stress tensors whereas $\delta S^{(2)}$ of 2-point functions. In turns out that these integrals can again be done with considerable effort on $\mathbb{S}^1 \times \mathbb{H}_{d-1}$. It is $\delta S^{(2)}$ that yields the result we mentioned above. In a surprising turn of events the modular Hamiltonian term $\delta S^{(1)}$ does not contribute to the non-local part of the entanglement density. It's cancellation though is highly non-trivial: The terms surviving the regularization turn out to be proportional to a combination of parameters which vanishes by conservation constraints on the stress-tensor.

5 Conclusion

In this note we gave a quick accounting of the state of perturbation theory on entanglement quantities. We went into some amount detail on the perturbation of the vacuum modular Hamiltonian for a half-space showing how its perturbations are related to the ANEC. We also gave a summary of the papers dealing with both operator deformations, which once again hint at a close relationship between gravity and entanglement even outside of the holographic regime and shape deformations of entanglement entropy, which give insights into universal contributions to entanglement entropy like corner contributions or deformations across spheres.

It should be fairly obvious from all of the above that a lot this subject is still mysterious and left to be explored, but it also seems evident that there are a lot of potentially interesting results which can come out of this study. In this note we talked about a proof of the ANEC, but these results and their cousins have also been used to prove other results like for example c-theorems[22].

References

- [1] G. 't Hooft, “Dimensional reduction in quantum gravity,” in *Salamfest 1993:0284-296*, pp. 0284–296, 1993.
- [2] L. Susskind, “The World as a hologram,” *J. Math. Phys.*, vol. 36, pp. 6377–6396, 1995.
- [3] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Int. J. Theor. Phys.*, vol. 38, pp. 1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)].
- [4] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.*, vol. 2, pp. 253–291, 1998.
- [5] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” *JHEP*, vol. 08, p. 045, 2006.
- [6] V. E. Hubeny, M. Rangamani, and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” *JHEP*, vol. 07, p. 062, 2007.
- [7] T. Faulkner, R. G. Leigh, O. Parrikar, and H. Wang, “Modular Hamiltonians for Deformed Half-Spaces and the Averaged Null Energy Condition,” *JHEP*, vol. 09, p. 038, 2016.
- [8] T. Faulkner, R. G. Leigh, and O. Parrikar, “Shape Dependence of Entanglement Entropy in Conformal Field Theories,” *JHEP*, vol. 04, p. 088, 2016.
- [9] T. Faulkner, “Bulk Emergence and the RG Flow of Entanglement Entropy,” *JHEP*, vol. 05, p. 033, 2015.
- [10] M. Rangamani and T. Takayanagi, “Holographic Entanglement Entropy,” *Lect. Notes Phys.*, vol. 931, pp. pp.–, 2017.
- [11] M. Van Raamsdonk, “Lectures on Gravity and Entanglement,” in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015*, pp. 297–351, 2017.
- [12] P. V. Buividovich and M. I. Polikarpov, “Entanglement entropy in gauge theories and the holographic principle for electric strings,” *Phys. Lett.*, vol. B670, pp. 141–145, 2008.
- [13] W. Donnelly, “Decomposition of entanglement entropy in lattice gauge theory,” *Phys. Rev.*, vol. D85, p. 085004, 2012.
- [14] H. Casini, M. Huerta, and J. A. Rosabal, “Remarks on entanglement entropy for gauge fields,” *Phys. Rev.*, vol. D89, no. 8, p. 085012, 2014.
- [15] W. Donnelly and A. C. Wall, “Entanglement entropy of electromagnetic edge modes,” *Phys. Rev. Lett.*, vol. 114, no. 11, p. 111603, 2015.
- [16] W. Donnelly and A. C. Wall, “Geometric entropy and edge modes of the electromagnetic field,” *Phys. Rev.*, vol. D94, no. 10, p. 104053, 2016.

- [17] P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory: A Non-technical introduction,” *Int. J. Quant. Inf.*, vol. 4, p. 429, 2006.
- [18] P. Calabrese and J. Cardy, “Entanglement entropy and conformal field theory,” *J. Phys.*, vol. A42, p. 504005, 2009.
- [19] J. J. Bisognano and E. H. Wichmann, “On the Duality Condition for a Hermitian Scalar Field,” *J. Math. Phys.*, vol. 16, pp. 985–1007, 1975.
- [20] J. J. Bisognano and E. H. Wichmann, “On the Duality Condition for Quantum Fields,” *J. Math. Phys.*, vol. 17, pp. 303–321, 1976.
- [21] H. Casini, M. Huerta, and R. C. Myers, “Towards a derivation of holographic entanglement entropy,” *JHEP*, vol. 05, p. 036, 2011.
- [22] H. Casini, E. Teste, and G. Torroba, “The a-theorem and the Markov property of the CFT vacuum,” 2017.