



Monopoles, Instantons and Hopf Bundles

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1 Introduction

After decades of parallel development of quantum field theory by physicists on the one hand and the theory of fiber bundles by mathematicians on the other hand, the eventual collaboration of both sides yielded a plethora of deep and beautiful results in both fields. This paper is an effort to exhibit the very basics of the mathematical theory at the very basis of quantum field theory: Mathematical gauge theory. In this paper, we only superficially show the most basic relation between the two: we study an object of great importance to physicists, the magnetic monopole, and have a first glimpse at the profound relation between a crucial physical property called Dirac's quantization condition and the topology of the associated principal bundle.

This report is structured in maybe a bit unusual way. It has two parts: the first one is purely theoretical, whereas the second one is a number of applications of the theory presented in the first part. The main objects of this paper are complex and quaternionic principal bundles and their connections and the author feels like it is easier to understand these objects if their construction is presented in one continuous streak and not separated into little pieces as examples of theorems. This possibly induces a dearth of examples in the theoretical part but one can always jump ahead and see the corresponding example in the applications section.

In the first section, we set the background knowledge for things to come. We establish our notations and give a quick reminder of the theory of flows on differentiable manifolds. Then we quickly review Lie theory. More precisely, we state some basics about matrix Lie groups and then transition to the theory of the associated Lie algebras, which play a crucial role later on. In particular, we define some objects which are less common knowledge like the Cartan 1-form and the wedge product for vector-valued forms. We conclude this section with a rather terse but complete review of the very basics of homotopy theory. We review both the fundamental group and the higher homotopy groups, which will eventually give us a way to classify principal bundles over spheres.

The core of the text starts with the third section about bundles. In this section, we study the playing ground of gauge theory. We start with the most basic object which are locally trivial bundles, which in essence are spaces which locally are homeomorphic to a product space but not globally. In particular, we study lifting properties, which is to say we look at how an object in the base space determines a corresponding object in the bundle space and for example prove that there is no uniqueness for these, a problem which will be resolved later on with connections. We then quickly show how covering spaces fall into the category of locally trivial bundles with discrete fibers. The next step are principal bundles, which are basically locally trivial bundles with fibers which are diffeomorphic to a certain Lie group, which acts on the bundle in way which is consistent with the bundle structure. From this principal bundles onwards, most of the results are explicitly proven. We study some basic properties like bundle maps, equivalence of bundles and the reconstruction theorem, which gives a simple way of constructing principal bundles from simple objects called transition functions. Finally, we study an object, which plays an important role for connections, which is the fundamental vector field. The idea is to associate a vector field to every element of the Lie algebra of the structure group of the principal bundle.

The fourth section is concerned with the study of connections on principal bundles, which basically introduce a way of connecting fibers, playing a similar role than connections in Riemannian geometry, and induces a separation of the tangent space. But most importantly, it introduces the notion of gauge equivalence of bundles, culminating in the definition of moduli spaces, which eventually play the central role in mathematics with which we are not concerned in this paper, like Donaldson theory.

At this point, the theory part is concluded and we enter into the applications part. We start by determining the fundamental group of circle and giving an idea how to determine the higher homotopy groups of spheres. We then go on to proving one of the main results of this paper, which is a complete classification of principal bundles over spheres with pathwise connected structure groups by the higher homotopy groups of the corresponding spheres. We then proceed to construct which are of great interest to both physicists and mathematicians, which are the Hopf bundles, which basically are spheres seen as principal bundles over projective spaces. The most interesting objects are then the connections on Hopf bundles, because they are the objects which arise from the physical equations. We then have a little section which tries to illustrate very briefly the physical

motivation behind all of this and we conclude with a section presenting the directions in which to go next.

This report is entirely based on the book [1] by Naber. The author does not pretend to have created any original work in this paper. Every result and nearly every proof in this report can be found in that book.

Part I

The Theory

2 Background

2.1 Some basics and notations

2.1.1 Standard spaces and matrix groups

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Definition 2.1.1. Let $n \in \mathbb{N}^+$. Then $\mathbb{F}^n$ equipped with coordinatewise addition and right scalar multiplication forms a right module over $\mathbb{F}$. The standard basis $\{e_1, \dots, e_n\}$ is defined as $(e_i)^j = \delta_i^j$. We also define a bilinear, non-degenerate, definite-positive form:

$$\langle x, y \rangle = \sum_{i=1}^n x^i y^i$$

where $x = \sum_{i=1}^n x^i e_i$ and $y = \sum_{i=1}^n y^i e_i$.

Theorem 2.1.1.

Definition 2.1.2. Let $n \in \mathbb{N}^+$. The $\mathbb{F}$ -unitary group of order n is the set $U(n, \mathbb{F})$ of $n \times n$ matrices which preserve the bilinear form, which is equivalent to them being invertible and satisfying $A^{-1} = \bar{A}^T$. For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, they are called the orthogonal group $O(n)$, the unitary group $U(n)$ and the symplectic group $Sp(n)$.

We now state some standard results:

$$U(1) \simeq S^1 \quad \& \quad SU(2) \simeq Sp(1) \simeq S^3$$

Definition 2.1.3. Let S be the subspace of $\mathbb{F}^n$ defined by $S = \{x \in \mathbb{F}^n \mid \langle x, x \rangle = 1\}$.

Lemma 2.1.1. S is homeomorphic to either S^{n-1} , S^{2n-1} , or S^{4n-1} depending on whether $\mathbb{F}$ is $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, respectively.

We now exhibit an alternate representation of the matrix groups over $\mathbb{H}$. Let us identify $x = x^0 + x^1 i + x^2 j + x^3 k \in \mathbb{H}$ with $(z^1, z^2) = (x^0 + x^1 i, x^2 + x^3 i) \in \mathbb{C}^2$. We can then write any quaternionic matrix P as $P = A + B j$.

Definition 2.1.4. We define the map ϕ from the algebra of $n \times n$ quaternionic matrices to the algebra of $2n \times 2n$ complex matrices by:

$$\text{Let } P = A + B j \Rightarrow \phi(P) = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

One can then show the following properties.

Lemma 2.1.2. ϕ is an isomorphism of algebras that preserves the conjugate transpose, i.e. $\phi(\bar{P}^T) = \overline{\phi(P)}^T$. Furthermore, a $2n \times 2n$ complex matrix M has the form of $\phi(\cdot)$ if and only if

$$J M J^{-1} = \bar{M} \quad \text{where} \quad J = \begin{pmatrix} 0 \mathbb{1}_{n \times n} & \\ -\mathbb{1}_{n \times n} & 0 \end{pmatrix}$$

Using this lemma, we can give an alternate definition of the quaternionic matrix groups.

Definition 2.1.5. *We have the following quaternionic matrix groups:*

1. $GL(n, \mathbb{H}) = \{M \in GL(2n, \mathbb{C}) | JMJ^{-1} = \bar{M}\}$
2. $Sp(n) = \{M \in U(2n) | M^T JM = J\}$

2.1.2 Differentiable manifolds

At this point, we recall some basis results in the theory of flows and integral curves. For the rest of this section, X is a smooth manifold.

Definition 2.1.6. *Let $\mathbf{V}$ be a smooth vector field. A smooth curve $\alpha : \mathcal{I} \rightarrow X$ is an integral curve for $\mathbf{V}$ if its velocity vector at each point coincides with value of $\mathbf{V}$ at that point, i.e. if*

$$\alpha'(t) = \mathbf{V}(\alpha(t)) \quad \forall t \in \mathcal{I}$$

Theorem 2.1.2. *Let $\mathbf{V}$ be a smooth vector field on X . Then there exists an interval $(a(p), b(p)) \in \mathbb{R}$ and a smooth curve $\alpha_p : (a(p), b(p)) \rightarrow X$ such that*

1. $0 \in (a(p), b(p))$ and $\alpha_p(0) = p$
2. α_p is an integral curve for $\mathbf{V}$
3. α_p is the maximal integral curve of $\mathbf{V}$ through p at $t = 0$, i.e. if $(c, d) \ni 0$ and $\beta : (c, d) \rightarrow X$ is an integral curve of $\mathbf{V}$ with $\beta(0) = p$, then $(c, d) \subseteq (a(p), b(p))$ and $\beta = \alpha_p|_{(c, d)}$

Definition 2.1.7. *For every $t \in \mathbb{R}$, we define the set*

$$\mathcal{D}_t = \{p \in X | t \in (a(p), b(p))\}$$

and the map $\mathbf{V}_t : \mathcal{D}_t \rightarrow X$ by

$$\mathbf{V}_t(p) = \alpha_p(t)$$

Lemma 2.1.3. *Let $\mathbf{V} \in \mathcal{X}(X)$ and let $s, t \in \mathbb{R}$. Then the domain of $\mathbf{V}_s \circ \mathbf{V}_t$ is contained in $\mathcal{D}_{s+t}$ and equal to it if s and t have the same sign, and, on this domain, $\mathbf{V}_s \circ \mathbf{V}_t = \mathbf{V}_{s+t}$.*

Theorem 2.1.3. *Let $\mathbf{V} \in \mathcal{X}$. Then, for each real number t , $\mathcal{D}_t$ is an open set in X and $\mathbf{V}_t$ is a diffeomorphism of $\mathcal{D}_t$ onto $\mathcal{D}_{-t}$ with inverse $\mathbf{V}_{-t}$.*

Definition 2.1.8. *A vector field $\mathbf{V} \in \mathcal{X}$ is complete if $\mathcal{D}_t = X$ for all $t \in \mathbb{R}$.*

The following lemma is important because a great number of the standard matrix groups which feature in your run-of-the-mill gauge theories are compact.

Lemma 2.1.4. *If X is a compact, then any $\mathbf{V} \in \mathcal{X}$ is complete.*

Finally, we state for later convenience an alternative way of calculating the Lie bracket.

Lemma 2.1.5. *Let $\mathbf{V}, \mathbf{W} \in \mathcal{X}$, then:*

$$[\mathbf{V}, \mathbf{W}]_p = \lim_{t \rightarrow 0} \frac{(V_{-t})_* \alpha_p(t) (\mathbf{W}_{\alpha_p(t)}) - \mathbf{W}_p}{t}$$

2.1.3 Vector valued forms

Definition 2.1.9. *Let $\mathcal{V}$ be a vector space.*

1. *A $\mathcal{V}$ -valued 0-form is a map ϕ from X to $\mathcal{V}$. It may be written as $\phi = \phi^i e_i$ where the $\{e_i\}$ are a basis of $\mathcal{V}$ and the real-valued ϕ^i are called the components. ϕ is continuous, smooth, etc. if the components are.*

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2. A $\mathcal{V}$ -value 1-form is a map $\omega : TX \rightarrow \mathcal{V}$ i.e. if $p \in X$, then $\omega(p) = \omega_p : T_p(X) \rightarrow \mathcal{V}$. It may be written as $\omega = \omega^i e_i$, where each ω_i is an ordinary $\mathbb{R}$ -valued one-form. Again, ω is continuous, smooth, etc. if the components are.
 3. A $\mathcal{V}$ -valued 2-form is a map Ω which assigns to each $p \in X$ a bilinear map $\Omega_p : T_p(X) \times T_p(X) \rightarrow \mathcal{V}$. It may be written as $\Omega = \Omega^i e_i$, where the Ω_i are ordinary 2-forms. Again, ω is continuous, smooth, etc. if the components are.

Most operations are simply defined componentwise for vector valued forms. An exception here is the wedge product, which we now proceed to define.

Definition 2.1.10. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be finite-dimensional real vector spaces and that we are given a bilinear map $\rho : \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{W}$. Let ω be a $\mathcal{U}$ -valued form, η a $\mathcal{V}$ -valued form, then we define the ρ -wedge product by

$$(\omega \wedge_\rho \eta)_p(v, w) = \rho(\omega_p(v), \eta_p(w)) - \rho(\eta_p(v), \omega_p(w))$$

where $p \in X$ and $v, w \in T_p(X)$.

Lemma 2.1.6. If $\{u_i\}$ and $\{v_j\}$ are a basis for $\mathcal{U}$ and $\mathcal{V}$ respectively, then “basis expression” for the ρ -wedge product is

$$\omega \wedge_\rho \eta = \sum_{i=1}^d \sum_{j=1}^f (\omega^i \wedge \eta^j) \rho(u_i, v_j)$$

where $\omega = \omega^i u_i$ and $\eta = \eta^j v_j$.

Example 2.1.1. Let $\mathcal{U} = \mathcal{V} = \mathcal{W} = \text{Im}\mathbb{H}$ and take $\rho(x, y) = [x, y] = 2\text{Im}(xy)$. The previous lemma then gives

$$\omega \wedge_\rho \eta = 4(\omega^2 \wedge \eta^3 i + \omega^3 \wedge \eta^1 j + \omega^1 \wedge \eta^2 k)$$

where $\omega = \omega^1 i + \omega^2 j + \omega^3 k$ and similarly for η .

2.2 Some Lie theory

2.2.1 Matrix Lie groups

Definition 2.2.1. A Lie group is a differentiable manifold with a smooth group structure, i.e. $(x, y) \rightarrow xy$ and $x \rightarrow x^{-1}$ are smooth.

One can actually relax that definition, as the following lemma shows.

Lemma 2.2.1. Let G be a differentiable manifold that is also a group for which the group multiplication $(x, y) \rightarrow xy$ is a smooth map from $G \times G$ to G . Then G is Lie group.

Example 2.2.1. The real, complex and quaternionic numbers with their respective multiplications are Lie groups.

Example 2.2.2. The real, complex and quaternionic general linear groups are all Lie groups, as well as the classical groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ and $Sp(n)$.

Definition 2.2.2. Any subgroup of $GL(n, \mathbb{C})$ that is also a submanifold is called a matrix Lie group.

Example 2.2.3. Using the redefinition mentioned in 2.1.5, one can identify $GL(n, \mathbb{H})$ as a subgroup of $GL(2n, \mathbb{C})$.

All the classical groups are matrix Lie groups.

Lemma 2.2.2. Left and right multiplication by $g \in G$ where G is a Lie group, i.e. $L_g : G \rightarrow G$ defined by $L_g(h) = gh$ and similarly for other case, are diffeomorphisms from G onto G .

2.2.2 Lie algebras

Definition 2.2.3. A Lie algebra is a real vector space $\mathcal{L}$ on which is defined a bilinear operator $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ that is skew-symmetric and satisfies the Jacobi identity.

Example 2.2.4. Any set of matrices which is closed under the commutator is a Lie algebra when the bracket is identified with the commutator.

Every Lie group has associated to it a Lie algebra, which can be described in two different but equivalent ways, which we now briefly present. For the rest of section, let G be a Lie group.

Definition 2.2.4. Let $\mathbf{V}$ be a vector field in G (not necessarily smooth). Then $\mathbf{V}$ is left-invariant if

$$(L_g)_* \circ \mathbf{V} = \mathbf{V} \circ L_g \quad \forall g \in G \quad \Leftrightarrow \quad (L_g)_* \mathbf{V}_h = \mathbf{V}_{gh} \quad \forall g, h \in G$$

where L_g is left translation.

Lemma 2.2.3. A vector field $\mathbf{V}$ is left-invariant if and only if $(L_g)_* \mathbf{V}_e = \mathbf{V}_g$ for each $g \in G$.

Lemma 2.2.4. A left-invariant vector field $\mathbf{V}$ on G is smooth.

Lemma 2.2.5. The set $\mathcal{G}$ of all left-invariant vector fields on G is a linear subspace of $\mathcal{G}$ and the map $\mathbf{V} \rightarrow \mathbf{V}_e$ from $\mathcal{G}$ to $T_e(G)$ is an isomorphism. In particular, we have that $\dim \mathcal{G} = \dim T_e(G) = \dim G$.

Lemma 2.2.6. Let $\mathcal{G}$ be the vector space of left-invariant vector field of G and $\mathbf{V}, \mathbf{W} \in \mathcal{G}$. Then $[\mathbf{V}, \mathbf{W}] \in \mathcal{G}$.

So the Lie algebra $\mathcal{G}$ of a Lie group G can be described as:

1. The set of left-invariant vector fields with the standard Lie bracket
2. The tangent space to the identity $T_e(G)$ with a bracket defined in the following way: Let $v, w \in T_e(G)$ and $\mathbf{V}, \mathbf{W}$ the unique left-invariant vector fields which satisfy $\mathbf{V}_e = v$ and $\mathbf{W}_e = w$, then $[v, w] = [\mathbf{V}_e, \mathbf{W}_e]$

The point of this is given in the following theorem.

Theorem 2.2.1. For matrix Lie groups G , $T_{id}(G)$ can be identified with the set of velocity vectors at $t = 0$ to smooth (matrix) curves in G computed entry-wise. Furthermore, this set is closed under the standard matrix commutator and the Lie algebra $T_{id}(G)$ with this standard commutator is isomorphic to the abstract construction presented above.

Example 2.2.5. We give the Lie algebras of the classical matrix groups:

1. $\mathcal{GL}(n, \mathbb{F}) = M_n(\mathbb{F})$
2. $\mathfrak{o}(n) = \{M \in M_n(\mathbb{R}) | M^T = -M\}$
3. $\mathfrak{u}(n) = \{M \in M_n(\mathbb{C}) | \overline{M}^T = -M\}$
4. $\mathfrak{su}(n) = \{M \in M_n(\mathbb{C}) | \overline{M}^T = -M, \text{Tr} M = 0\}$
5. $\mathfrak{sp}(n) = \{M \in M_{2n}(\mathbb{C}) | \overline{M}^T = -M, JM + M^T J = 0\}$

Definition 2.2.5. Let $\mathcal{G}$ be a Lie algebra, $\{e_i\}$ be a basis for $\mathcal{G}$ and $\{\mathbf{e}_i\}$ be the unique corresponding left-invariant vector fields. Then

$$[e_i, e_j] = C_{ij}^k e_k \quad \Leftrightarrow \quad [\mathbf{e}_i, \mathbf{e}_j] = C_{ij}^k \mathbf{e}_k$$

where the equivalence follows by left-invariance. The constants C_{ij}^k are called the structure constants of the Lie algebra (relative to that basis).

We now just quickly recall some basic properties of the matrix exponential map.

Lemma 2.2.7. *The series*

$$\exp(A) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

converges absolutely and uniformly on any bounded region of $\mathcal{GL}(n, \mathbb{C})$ and we have the following properties:

1. $e^A e^B = e^{A+B} \quad \Leftrightarrow \quad [A, B] = 0$ where $A, B \in \mathcal{GL}(n, \mathbb{C})$
2. $\det A = e^{\text{Tr} A}$
3. The map $\exp : \mathcal{GL}(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$ defined by $A \rightarrow e^A$ is smooth
4. For any $A \in \mathcal{GL}(N, \mathbb{C})$, the map defined by $t \rightarrow e^{tA}$ is a smooth homomorphism from the additive group $\mathbb{R}$ to $GL(n, \mathbb{C})$ whose velocity vector at $t = 0$ is A
5. If we identify $T_0(\mathcal{GL}(n, \mathbb{C}))$ with $\mathcal{GL}(n, \mathbb{C})$ by the canonical isomorphism, then

$$\exp_{*0} : T_0(\mathcal{GL}(n, \mathbb{C})) \rightarrow \mathcal{GL}(n, \mathbb{C})$$

is the identity map

Corollary 2.2.1. $\exp : \mathcal{GL}(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$ is a diffeomorphism of some open neighborhood V of $0 \in \mathcal{GL}(n, \mathbb{C})$ onto an open neighborhood U of $\text{id} \in GL(n, \mathbb{C})$, i.e. $(U, (\exp|_V)^{-1})$ is a chart at $\text{id} \in GL(n, \mathbb{C})$.

Lemma 2.2.8. Let $\mathbf{A}$ be the element of $\mathcal{G}$ thought of as a left-invariant vector field and A the same element thought of as a complex matrix, so that one has $\mathbf{A}(\text{id}) = A$. Any $A \in \mathcal{G}$ can be thought of as $A = \alpha'(0)$ where $\alpha(t) = \exp(tA)$. Fix $g \in G$ and define $\alpha_g : \mathbb{R} \rightarrow G$ by $\alpha_g(t) = g \exp(tA)$. Then α_g is the unique maximal integral curve of $\mathbf{A}$ that starts at g , i.e. we have

$$\alpha_g(0) = g \quad \text{and} \quad \alpha'_g(t) = \mathbf{A}(\alpha_g(t)) \quad \forall t \in \mathbb{R}$$

This implies in particular that $\mathbf{A}$ is complete.

Definition 2.2.6. Let Θ be a 1-form on a Lie group G . Then Θ is left-invariant if for each $g \in G$, we have

$$(L_g)^* \Theta = \Theta$$

Lemma 2.2.9. Θ is left-invariant if and only if $\Theta_g(v) = \Theta_{\text{id}}((L_{g^{-1}})_{*} \text{id}(v))$ where $v \in T_g(G)$

Lemma 2.2.10. If Θ is a left-invariant 1-form on G , then it is smooth.

Definition 2.2.7. Let G be a matrix Lie group and $\mathcal{G}$ its Lie algebra. The Cartan canonical 1-form is the $\mathcal{G}$ -valued 1-form Θ on G defined as

$$\Theta_g(v) = (L_{g^{-1}})_{*} v \quad \forall g \in G, \forall v \in T_g(G) \quad \Leftrightarrow \quad \Theta_g(A(g)) = A(\text{id})$$

where A is a left-invariant vector field on G .

Lemma 2.2.11. Let $\{e_i\}$ be a basis for $\mathcal{G}$ and $\{\Theta^i\}$ be the unique left-invariant $\mathbb{R}$ -valued 1-forms on G which satisfy $\Theta^i(e_j) = \delta_j^i$. Then the Cartan canonical 1-form is given by $\Theta = \sum_i \Theta^i e_i$.

Definition 2.2.8. A $\mathcal{G}$ -valued 1-form is right equivariant if

$$(R_g)^* \omega = \text{ad}_{g^{-1}} \circ \omega \quad \forall g \in G$$

Lemma 2.2.12. The Cartan canonical 1-form is left-invariant and right equivariant.

Lemma 2.2.13. Let H be a subgroup of G which is also a submanifold and $i : H \hookrightarrow G$ the inclusion map, then the Cartan 1-form Θ_H of H is the restriction to H of the Cartan 1-form Θ_G of G , i.e.

$$\Theta_H = i^* \Theta_G$$

Example 2.2.6. We give the Cartan 1-form of $GL(n, \mathbb{R})$. $GL(n, \mathbb{R})$ is an open submanifold of $\mathbb{R}^{n^2}$ with standard coordinate entry functions x^{ij} where $i, j = 1, \dots, n$ and $\{\frac{\partial}{\partial x^{ij}}\}_{i,j=1,\dots,n}$ is a basis for $\mathcal{GL}(n, \mathbb{R})$. The Cartan 1-form is then given by

$$\Theta_g = (x^{ik}(g^{-1})dx^{kj}(g)) \frac{\partial}{\partial x^{ij}}|_{id}$$

This can formally be identified with a matrix product

$$\Theta_g = g^{-1}dx(g)$$

where $dx(g)_{ij} = dx^{ij}(g)$.

Lemma 2.2.14. The Cartan canonical 1-form satisfies the following equations for $k = 1, \dots, n$

$$d\Theta^k = -\frac{1}{2}C_{ij}^k \Theta^i \wedge \Theta^j = -\sum_{i < j} C_{ij}^k \Theta^i \wedge \Theta^j$$

called the Maurer-Cartan equation

Proof. Since $\Theta^i \wedge \Theta^j$ evaluated at id is the standard basis for 2-forms (wedge product of the dual basis and by left-invariance of the Cartan 1-form, it is sufficient to prove

$$d\Theta^k(\mathbf{e}_i, \mathbf{e}_j) = -C_{ij}^k$$

Let $\mathbf{V}, \mathbf{W}$ be left-invariant vector field, then $\Theta\mathbf{V}$ and $\Theta\mathbf{W}$ are constant. So that

$$\begin{aligned} d\Theta(\mathbf{V}, \mathbf{W}) &= \mathbf{V}(\Theta\mathbf{W}) - \mathbf{W}(\Theta\mathbf{V}) - \Theta([\mathbf{V}, \mathbf{W}]) = -\Theta([\mathbf{V}, \mathbf{W}]) \\ \Rightarrow d\Theta^k(\mathbf{e}_i, \mathbf{e}_j) &= -\Theta^k(C_{ij}^l e_j) = -C_{ij}^k \end{aligned}$$

□

Lemma 2.2.15. If we define the wedge product of $\mathcal{G}$ -valued forms via the natural bracket $\rho(x, y) \equiv [x, y]$ where $x, y \in G$. Then the Cartan 1-form also satisfies the following equation

$$d\Theta + \frac{1}{2}[\Theta, \Theta] = 0$$

which is called the equation of structure of G .

Definition 2.2.9. Let G be a matrix group and $g \in G$ fixed. The map $Ad_g : G \rightarrow G$ defined by

$$Ad_g(h) = ghg^{-1} \quad \forall h \in h$$

The adjoint representation is the derivative is the derivative of Ad_g at the identity, i.e.

$$ad_g = (Ad_g)_{*id} = (L_g)_{*g^{-1}} \circ (R_{g^{-1}})_{*id} = (R_{g^{-1}})_{*g} \circ (L_g)_{*id}$$

Lemma 2.2.16. The map $g \rightarrow ad_g$ is a representation, i.e. it is a homomorphism of G into the group of nonsingular linear transformation on the Lie algebra $\mathcal{G}$.

Proof. By composition of diffeomorphisms, Ad_g is a diffeomorphism. Furthermore, since $Ad_g(id) = id$, ad_g, ad_g carries $\mathcal{G}$ isomorphically onto $\mathcal{G}$. Linearity comes from the Linearity of the derivative. Finally, a trivial calculation shows that

$$ad_{gh} = ad_g \circ ad_h$$

which proves the homomorphism property. □

Lemma 2.2.17. Let G be a matrix Lie group, $\mathcal{G}$ its Lie algebra and $g \in G$. Then for each $A \in \mathcal{G}$, $gAg^{-1} \in \mathcal{G}$ and the isomorphism $ad_g : \mathcal{G} \rightarrow \mathcal{G}$ is given by $ad_g(A) = gAg^{-1}$.

2.3 Homotopy

2.3.1 The fundamental group

Definition 2.3.1. Let X be a topological space. A path in X (from x_0 to x_1) is a continuous map $\alpha : [0, 1] \rightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. If $x_0 = x_1$, α is called a loop.

Definition 2.3.2. Let α, β be two paths such that $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$. α and β are path homotopic or homotopic relative to $0, 1$ if there exists a continuous map $F : [0, 1] \times [0, 1] \rightarrow X$, called a path homotopy from α to β in X which satisfies the following conditions:

1. $F(0, t) = x_0$ and $F(1, t) = x_1 \quad \forall t \in [0, 1]$
2. $F(s, 0) = \alpha(s)$ and $F(s, 1) = \beta(s) \quad \forall s \in [0, 1]$

Furthermore, we define for each $t \in [0, 1]$ a path from x_0 to x_1 given by $F_t(s) \equiv F(s, t)$.

If α and β are path homotopic, we denote this by $\alpha \simeq \beta \text{ rel } \{0, 1\}$.

Lemma 2.3.1. The notion of path homotopy defines an equivalence relation on the set of paths in X from x_0 to x_1 . More precisely, let X be a topological space and α, β, γ paths in X from x_0 to x_1 . Then, we have the following properties:

1. $\alpha \simeq \alpha \text{ rel } \{0, 1\}$
2. $\alpha \simeq \beta \text{ rel } \{0, 1\} \Rightarrow \beta \simeq \alpha \text{ rel } \{0, 1\}$
3. $\alpha \simeq \beta \text{ rel } \{0, 1\}$ and $\beta \simeq \gamma \text{ rel } \{0, 1\} \Rightarrow \alpha \simeq \gamma \text{ rel } \{0, 1\}$

Lemma 2.3.2. Let α, β be paths in X from x_0 to x_1 and x_1 to x_2 respectively and $[\alpha]$ and $[\beta]$ be their equivalence classes under the path homotopy equivalence relation, called path homotopy classes. We can define two operations on the path homotopy classes.

1. Inversion: We define $\alpha^\leftarrow : [0, 1] \rightarrow X$ called “ α backwards” by $\alpha^\leftarrow(s) = \alpha(1 - s)$. This operation is well-defined on path homotopy classes in the sense that if α, α' are paths in X from x_0 to x_1 such that $\alpha \simeq \alpha' \text{ rel } \{0, 1\}$, then $\alpha^\leftarrow \simeq (\alpha')^\leftarrow \text{ rel } \{0, 1\}$. We can then unambiguously define $[\alpha]^{-1} = [\alpha]^\leftarrow = [\alpha^\leftarrow]$.
2. Product: We define $\alpha\beta : [0, 1] \rightarrow X$ called “ α followed by β ” by

$$\alpha\beta(s) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \beta(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

This operation is well-defined on path homotopy classes in the sense that if α, α' are paths in X from x_0 to x_1 and β, β' are paths in X from x_1 to x_2 such that $\alpha \simeq \alpha' \text{ rel } \{0, 1\}$ and $\beta \simeq \beta' \text{ rel } \{0, 1\}$, then $\alpha\beta \simeq \alpha'\beta' \text{ rel } \{0, 1\}$. We can then unambiguously define $[\alpha][\beta] = [\alpha\beta]$.

3. Constant loop: We denote by x_0 the trivial loop at x_0 which is simply defined as $\alpha(s) = x_0$ for all $s \in [0, 1]$.

Having defined these operations, we now obtain the following crucial theorem.

Theorem 2.3.1. Let X be a topological space and x_0 a point in X . Let $\pi_1(X, x_0)$ be the set of all path homotopy classes of loops at x_0 in X . Then $\pi_1(X, x_0)$ equipped with the operations defined in the previous lemma forms a group, called the fundamental group, or first homotopy group, of X at the base point x_0 .

We now proceed to state the theorem which specifies under which conditions the reference to the base point can be omitted.

Theorem 2.3.2. Let x_0 and x_1 be two points in a topological space X and σ a path in X from x_0 to x_1 . Then the map

$$\sigma_\# : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1) \text{ defined by } \sigma_\#([\alpha]) = [\sigma\alpha\sigma^\leftarrow] \quad \forall [\alpha] \in \pi_1(X, x_0)$$

is an isomorphism.

Corollary 2.3.1. *If X is pathwise connected, then*

$$\pi_1(X, x_0) \simeq \pi_1(X, x_1) \quad \forall x_0, x_1 \in X$$

Definition 2.3.3. *Let X be a topological space and x_0 a point in X . Then the pair (X, x_0) is called a pointed space with base point x_0 . Let (X, x_0) and (Y, y_0) be two pointed spaces. Then a map of pointed spaces is a map $f : (X, x_0) \rightarrow (Y, y_0)$ such that $f(x_0) = y_0$, i.e. it is a map that preserves the base points.*

Theorem 2.3.3. *Let (X, x_0) and (Y, y_0) be pointed spaces and f a map of pointed spaces. Then f induces a homomorphism*

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0) \text{ defined by } f_*([\alpha]) = [f \circ \alpha] \quad \forall [\alpha] \in \pi_1(X, x_0)$$

Furthermore, we have the following properties:

1. *If $(Y, y_0) = (X, x_0)$ and $f = id_X$, then $f_* = (id_X)_* = id_{\pi_1(X, x_0)}$*
2. *If (Z, z_0) is another pointed space and $g : (Y, y_0) \rightarrow (Z, z_0)$ another map, then $(g \circ f)_* = g_* \circ f_*$*

Definition 2.3.4. *Let X, Y be topological spaces and A a subset of X and $f, g : X \rightarrow Y$ two continuous maps which coincide on A , i.e. such that $f|_A = g|_A$. Then f is homotopic to g relative to A if there exists a continuous map $F : X \times [0, 1] \rightarrow Y$ which satisfies the following conditions:*

1. $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$
2. $F(a, t) = f(a) = g(a)$ for all $t \in [0, 1]$ and $a \in A$

If such a map exists, we write $f \simeq g \text{ rel } A$. If A is the empty set, we say that f and g are homotopic and we write $f \simeq g$.

Definition 2.3.5. *A pathwise connected space X is simply connected if its fundamental group is the trivial group, i.e. $\pi_1(X) = 0$.*

Lemma 2.3.3. *Let X and Y be topological spaces, A a subset of X and $f, g, h : X \rightarrow Y$ continuous maps such that $f|_A = g|_A = h|_A$. Then homotopy relative to A is an equivalence relation, i.e. we have the following properties:*

1. $f \simeq f \text{ rel } A$
2. $f \simeq g \text{ rel } A \Rightarrow g \simeq f \text{ rel } A$
3. $f \simeq g \text{ rel } A \text{ and } g \simeq h \text{ rel } A \Rightarrow f \simeq h \text{ rel } A$

Definition 2.3.6. *The equivalence classes of the homotopy equivalence relation are called homotopy classes relative to A . If $A = \emptyset$, then the set of all homotopy classes of maps from X to Y is denoted $[X, Y]$. In the same way, if (X, x_0) and (Y, y_0) are pointed spaces, then the set of homotopy classes relative to $\{x_0\}$ of maps $f : (X, x_0) \rightarrow (Y, y_0)$ is denoted by $[(X, x_0), (Y, y_0)]$.*

Definition 2.3.7. *A continuous map $f : X \rightarrow Y$ where X, Y are topological spaces is nullhomotopic, if it is homotopic to some constant map of X into Y .*

Definition 2.3.8. *A topological space X is contractible if the identity map is nullhomotopic.*

Lemma 2.3.4. *A topological space Y is contractible if and only if any maps into it are homotopic, i.e. if for any topological space X , any two maps $f, g : X \rightarrow Y$ are homotopic.*

Lemma 2.3.5. *Let Y be a contractible topological space. Then any continuous map $f : Y \rightarrow X$ is nullhomotopic.*

Theorem 2.3.4. *A contractible space is simply connected.*

Lemma 2.3.6. *Let X be a topological space and $x_0 \in X$ and let $p_0 \in S^1$. Then there is a one-to-one correspondence between $\pi_1(X, x_0)$ and $[(X, x_0), (S^1, p_0)]$.*

We define a new equivalence relation.

Definition 2.3.9. Let X and Y be topological spaces. X and Y are homotopically equivalent (or of the same homotopy type) if there exists a continuous map $h : X \rightarrow Y$ called a homotopy equivalence, which means that there exists another continuous map $h' : Y \rightarrow X$ such that $h' \circ h \simeq id_X$ and $h' \circ h \simeq id_Y$.

We then obtain the following theorem.

Theorem 2.3.5. Let $f : (X, x_0) \rightarrow (Y, y_0)$ and $g : (X, x_0) \rightarrow (Y, y_1)$ be homotopic maps. Let $\sigma : [0, 1] \rightarrow Y$ be the path from y_0 to y_1 given by $\sigma(t) = F(x_0, t)$. We then have $f_* = \sigma_{\#} \circ g_*$. Moreover, f_* is an isomorphism if and only if g_* is an isomorphism.

An important corollary is that the fundamental is actually a homotopy invariant.

Corollary 2.3.2. Let $h : X \rightarrow Y$ be a homotopy equivalence. Then, for every $x_0 \in X$, $h_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, h(x_0))$ is an isomorphism. In particular, if X and Y are pathwise connected spaces of the same homotopy type, then their fundamental groups are isomorphic.

Lemma 2.3.7. Let X be a topological space, $n \leq 1$ and $f : S^n \rightarrow X$ a continuous map. Then f is nullhomotopic if and only if it has a continuous extension to the disc D^{n+1} , i.e. there exists a continuous map $\tilde{f} : D^{n+1} \rightarrow X$ such that $\tilde{f}|_{\partial D^{n+1}} = \tilde{f}|_{S^n} = f$.

Definition 2.3.10. Let A be a subset of a topological space X . Then A is a retract of X if there exists a continuous map $r : X \rightarrow A$ called a retraction such that $r|_A = id_A$.

Lemma 2.3.8. If A is a retract of X , then, for any $a \in A$, $\pi_1(A, a)$ is isomorphic to a subgroup of $\pi_1(X, a)$.

Definition 2.3.11. Let A be a subset of a topological space X . Then A is a deformation retract of X if there exists a homotopy $F : X \times [0, 1] \rightarrow X$, called a deformation retraction, such that $F_0 = id_X$ and $F_1 : X \rightarrow X$ satisfies $F_1(x) \in A \quad \forall x \in X$ and $F_1(a) = a \quad a \in A$.

Theorem 2.3.6. Let X and Y be topological spaces with $x_0 \in X$ and $y_0 \in Y$. Then

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

2.3.2 Higher homotopy groups

Definition 2.3.12. Let $n \in \mathbb{N}$. An n -cube in $\mathbb{R}^n$ is the set $I^n = [0, 1]^n = \{(s^1, \dots, s^n) \in \mathbb{R}^n | 0 \leq s^i \leq 1, \forall i \in \{1, \dots, n\}\}$. The boundary ∂I^n of I^n consists of all $(s^1, \dots, s^n) \in I^n$ for which $s^i = 0$ or $s^i = 1$ for at least one value of i .

Definition 2.3.13. Let X be a topological space, $x_0 \in X$ and $n \in \mathbb{N}$. An n -loop at x_0 is a continuous map $\alpha : I^n \rightarrow X$ such that $\alpha(\partial I^n) = \{x_0\}$.

Lemma 2.3.3 proves that this defines an equivalence relation.

Lemma 2.3.9. Let α, β be n -loops in X at x_0 and $[\alpha]$ and $[\beta]$ be their equivalence classes under the homotopy equivalence relation (relative to ∂I^n), called homotopy classes. We can define two operations on the path homotopy classes.

1. Inversion: We define $\alpha^{\leftarrow} : I^n \rightarrow X$ called “ α backwards” by $\alpha^{\leftarrow}(s^1, \dots, s^n) = \alpha(1 - s^1, s^2, \dots, s^n)$. This operation is well-defined on path homotopy classes in the sense that if α, α' are n -loops at x_0 such that $\alpha \simeq \alpha' \text{ rel } \partial I^n$, then $\alpha^{\leftarrow} \simeq (\alpha')^{\leftarrow} \text{ rel } \partial I^n$. We can then unambiguously define $[\alpha]^{-1} = [\alpha]^{\leftarrow} = [\alpha^{\leftarrow}]$.

2. Sum: We define $\alpha + \beta : I^n \rightarrow X$ called “ α followed by β ” by

$$(\alpha + \beta)(s^1, \dots, s^n) = \begin{cases} \alpha(2s^1, s^2, \dots, s^n) & \text{if } 0 \leq s^1 \leq \frac{1}{2} \\ \beta(2s^1 - 1, s^2, \dots, s^n) & \text{if } \frac{1}{2} \leq s^1 \leq 1 \end{cases}$$

This operation is well-defined on path homotopy classes in the sense that if $\alpha, \alpha', \beta, \beta'$ are n -loops at x_0 such that $\alpha \simeq \alpha' \text{ rel } \partial I^n$ and $\beta \simeq \beta' \text{ rel } \partial I^n$, then $\alpha + \beta \simeq \alpha' + \beta' \text{ rel } \partial I^n$. We can then unambiguously define $[\alpha] + [\beta] = [\alpha + \beta]$.

3. *Constant loop:* We denote by x_0 the trivial loop at x_0 which is simply defined as $\alpha(s) = x_0$ for all $s \in [0, 1]$.

Theorem 2.3.7. *Let X be a topological space, x_0 a point in X and $n \geq 2 \in \mathbb{N}$. Let $\pi_n(X, x_0)$ be the set of all homotopy classes (relative to ∂I^n) of n -loops at x_0 in X . Then $\pi_n(X, x_0)$ equipped with the operations defined in the previous lemma forms an Abelian group, called the n^{th} homotopy group of X at the base point x_0 .*

The higher homotopy groups satisfy mostly the same properties than the fundamental group, even if the proofs are somewhat more cumbersome.

Theorem 2.3.8. *Let (X, x_0) and (Y, y_0) be pointed spaces and f a map of pointed spaces. Then for each $n \geq 2$, f induces a homomorphism*

$$f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0) \text{ defined by } f_*([\alpha]) = [f \circ \alpha] \quad \forall [\alpha] \in \pi_n(X, x_0)$$

Furthermore, we have the following properties:

1. If $(Y, y_0) = (X, x_0)$ and $f = \text{id}_X$, then $f_* = (\text{id}_X)_* = \text{id}_{\pi_n(X, x_0)}$
2. If (Z, z_0) is another pointed space and $g : (Y, y_0) \rightarrow (Z, z_0)$ another map, then $(g \circ f)_* = g_* \circ f_*$

We now state theorem 2.3.2 for higher homotopy groups. Notice that the result is the same, but the actual from of the isomorphism $\sigma|_{\#}$ is less straightforward.

Theorem 2.3.9. *Let x_0 and x_1 be points in the topological space X and suppose there exists a path $\sigma : [0, 1] \rightarrow X$ in X from x_0 to x_1 . For each $n \geq 2$, one can then define a map $\sigma|_{\#} : \pi_n(X, x_0) \rightarrow \pi_n(X, x_1)$ as follows:*

Fix $[\alpha] \in \pi_n(X, x_0)$ and select a homotopy $F : I^n \times [0, 1] \rightarrow X$ with $F_0(s) = \alpha(s)$ for $s \in I^n$ and for $s \in \partial I^n$, $F(s, t) = \sigma^{\leftarrow}(s)$ for all $t \in [0, 1]$. Then set $\sigma|_{\#}([\alpha]) = [F_1]$.

This map $\sigma|_{\#}$ is a well-defined isomorphism that depends only on the path homotopy class $[\sigma]$ of σ , i.e. $\sigma \simeq \sigma'$ rel $\{0, 1\}$, then $(\sigma')|_{\#} = \sigma|_{\#}$.

With the $\sigma|_{\#}$ defined as above, the statement of theorem 2.3.5 is unchanged.

Theorem 2.3.10. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ and $g : (X, x_0) \rightarrow (Y, y_1)$ be homotopic maps. Let $\sigma : [0, 1] \rightarrow Y$ be the path from y_0 to y_1 given by $\sigma(t) = F(x_0, t)$. We then have for each $n \geq 2$ $f_* = \sigma|_{\#} \circ g_*$. Moreover, f_* is an isomorphism if and only if g_* is an isomorphism.*

Corollary 2.3.3. *Let $h : X \rightarrow Y$ be a homotopy equivalence. Then, for every $x_0 \in X$ and every $n \geq 2$, $h_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, h(x_0))$ is an isomorphism.*

Corollary 2.3.4. *Let X be a contractible space. Then, for any $n \geq 1$, $\pi_n(X) = 0$.*

We now state lemma 2.3.6 for higher homotopy groups and a reformulation of this lemma in terms of maps on spheres, which is crucial for understanding the higher homotopy groups of spheres.

Lemma 2.3.10. *Let X be a topological space and $x_0 \in X$ and let $p_0 \in S^n$ for $n \geq 2$. Then there is a one-to-one correspondence between $\pi_n(X, x_0)$ and $[(S^n, p_0), (X, x_0)]$.*

Lemma 2.3.11. *Let X be a topological space, $x_0 \in X$ and γ a loop at x_0 in X such that $\gamma \simeq x_0$ rel $\{0, 1\}$. Let $n \in \mathbb{N}$, $p_0 \in S^n$ and $G : S^n \times [0, 1] \rightarrow X$ a homotopy such that $G(p_0, t) = \gamma(t)$ for all $t \in [0, 1]$. Then $G_0 \simeq G_1$ rel $\{p_0\}$.*

A remarkable consequence of this lemma is that when X is simply connected, the relative homotopies that go into the definition of $[(S^n, p_0), (X, x_0)]$ can be replace by free homotopies.

Theorem 2.3.11. *Let X be simply connected and $x_0 \in X$. Let n be a positive integer, $p_0 \in S^n$ and $f, g : (S^n, p_0) \rightarrow (X, x_0)$. Then f and g are homotopic relative to p_0 if and only if they are freely homotopic*

3 Bundles

3.1 Locally trivial bundles

Definition 3.1.1. A locally trivial bundle $(P, X, \mathcal{P}, Y)$ consists of:

1. A topological space P called the bundle space or total space
2. A Hausdorff space X called the base space
3. A continuous map $\mathcal{P} : P \rightarrow X$ of P onto X called the projection
4. A Hausdorff space Y called the fiber
5. A local trivialization: $\forall x_0 \in X$, there exists an open set $V \subset X$ such that $x_0 \in V$ and a homeomorphism

$$\Phi : V \times Y \rightarrow \mathcal{P}^{-1}(V) \subset P \quad \text{such that} \quad \mathcal{P} \circ \Phi(x, y) = x$$

The neighborhood V is called a locally trivializing neighborhood and the pair (V, Φ) is the local trivialization.

Example 3.1.1. Let X and Y be Hausdorff spaces and define $P = X \times Y$. Let $\mathcal{P} : P \rightarrow X$ be the projection on the first factor, i.e. $\mathcal{P}(p) = \mathcal{P}(x, y) = x$ and take (X, id_X) to be the local trivialization. Then $(P, X, \mathcal{P}, Y)$ is a locally trivial bundle called the trivial bundle or the product bundle.

Example 3.1.2. Let $P = \mathbb{R}$, $X = S^1$ and $\mathcal{P}(s) = e^{2\pi si}$ is a locally trivial bundle with fiber $\mathbb{Z}$.

Example 3.1.3. Hopf bundles (see section 7).

The following lemma state some relatively simple consequences of the definition.

Lemma 3.1.1. For all $x \in X$, $\mathcal{P}^{-1}(x)$ is homeomorphic to Y .

Proof. Let (V, Φ) be a local trivialization about x . If $p \in \mathcal{P}^{-1}(x)$, we have that

$$\Phi^{-1}(p) = (x, y')$$

for some $y' \in Y$. Indeed, we have:

$$\Phi^{-1}(p) = (x', y') \Rightarrow p = \Phi(x', y') \Rightarrow \mathcal{P}(p) = x' \Rightarrow x = x'$$

We define the maps $\pi_2 : V \times Y \rightarrow Y$ by $\pi_2(x, y) = y$ and $u_x : Y \rightarrow \{x\} \times Y$ by $u_x(y) = (x, y)$ which are clearly continuous. Then we have:

$$\psi \equiv \pi_2 \circ \Phi^{-1} : \mathcal{P}^{-1}(x) \rightarrow Y$$

This is defined on all of $\mathcal{P}^{-1}(x)$. Its inverse is

$$\psi^{-1} \equiv \Phi \circ u_x : Y \rightarrow \mathcal{P}^{-1}(x)$$

So ψ is a homeomorphism. □

Lemma 3.1.2. Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle. Then $\mathcal{P} : P \rightarrow X$ is a continuous, open surjection.

Lemma 3.1.3. Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle. Then bundle space P is a Hausdorff space.

Lemma 3.1.4. Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle. Then X is homeomorphic to $P/\sim$ where $\sim$ is the equivalence relation which identifies points in the same fiber of $\mathcal{P}$, i.e. $p_1 \sim p_2 \Leftrightarrow \mathcal{P}(p_1) = \mathcal{P}(p_2)$.

Proof. We define $G : P \rightarrow P/\sim$ by $G(p) = [p]$ and $\bar{f} : P/\sim \rightarrow X$ by $\bar{f}([p]) = \mathcal{P}(p)$. This is onto since $\mathcal{P}$ is onto and one-to-one since

$$\bar{f}([p_1]) = \bar{f}([p_2]) \Rightarrow p_1 \simeq p_2 \Rightarrow [p_1] = [p_2]$$

and $\bar{f}$ is continuous by composition. Furthermore, we have that $(\bar{f})^{-1}$ defined by

$$(\bar{f})^{-1}(x) = G(\mathcal{P}^{-1}(x))$$

which is continuous by the definition of the quotient topology, so that $\bar{f}$ is a homeomorphism. $\square$

Lemma 3.1.5. *Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle with connected base space and connected fiber. Then the bundle space is connected.*

Lemma 3.1.6. *Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle with compact base space and compact fiber. Then the bundle space is compact.*

Definition 3.1.2. *Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle and $f : Z \rightarrow X$ where Z is some topological space. A lift of f to P is a continuous map $\tilde{f} : Z \rightarrow P$ such that $\mathcal{P} \circ \tilde{f} = f$.*

Example 3.1.4. *In general, lifts do not have to exist. Take for instance the locally trivial bundle $(\mathbb{R}, S^1, \mathcal{P}, \mathbb{Z})$. There exists no continuous map $\tilde{f} : S^1 \rightarrow \mathbb{R}$ such that $\mathcal{P} \circ \tilde{f} = \text{id}$. Suppose there exists such a map. Then the image $\tilde{f}(S^1)$ is compact and connected and so it is a closed, bounded subinterval $[a, b]$ of $\mathbb{R}$. Now, $\tilde{f}$ cannot not be one-to-one since otherwise it would be a homeomorphism (since a continuous bijection out of a compact space is a homeomorphism) and S^1 is not homeomorphic to S^1 . Thus, there exists points $x_0, x_1 \in S^1$ with $x_0 \neq x_1$, but $\tilde{f}(x_0) = \tilde{f}(x_1)$, which would in turn imply that $\mathcal{P} \circ \tilde{f}(x_0) = \mathcal{P} \circ \tilde{f}(x_1)$ so that $x_0 = x_1$ which is a contradiction.*

Definition 3.1.3. *Let V be an open set in the base space X of a locally trivial bundle. Then a local (smooth) cross-section of the bundle on V is a continuous (smooth) map $s : V \rightarrow P$ of V into the bundle space P such that $\mathcal{P} \circ s = \text{id}_V$, i.e. it is a continuous selection of an element in each fiber above V .*

Theorem 3.1.1 (The Path Lifting Theorem). *Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle and α a path in the base space X . Then, for any point p in the fiber $\mathcal{P}^{-1}(\alpha(0))$ above $\alpha(0)$, there exists a lift $\tilde{\alpha} : [0, 1] \rightarrow P$ of α to P with $\tilde{\alpha}(0) = p$.*

Corollary 3.1.1. *Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle with pathwise connected base space and pathwise connected fiber. Then the bundle space is pathwise connected.*

Theorem 3.1.2 (Homotopy Lifting Theorem). *Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle and $n \in \mathbb{N}^+$. Suppose $f : I^n \rightarrow X$ is a continuous map that lifts to $\tilde{f} : I^n \rightarrow P$ and $F : I^n \times [0, 1] \rightarrow X$ is a Homotopy with $F(x, 0) = f(x)$ for each $x \in X$. Then there exists a Homotopy $\tilde{F} : I^n \times [0, 1] \rightarrow P$ such that $\mathcal{P} \circ \tilde{F} = F$ and $\tilde{F}(x, 0) = \tilde{f}(x)$ for every $x \in X$.*

It turns out that one of the hypothesis of the theorem is superfluous, as the following corollary states.

Corollary 3.1.2. *Let $(P, X, \mathcal{P}, Y)$ be a locally trivial bundle and $n \in \mathbb{N}^+$. Then any continuous map $f : I^n \rightarrow X$ lifts to a continuous map $\tilde{f} : I^n \rightarrow P$ with $\mathcal{P} \circ \tilde{f} = f$.*

Proof. I^n is convex and therefore contractible. By Lemma 2.3.5, f is nullhomotopic. Let $F : I^n \times [0, 1] \rightarrow X$ be a homotopy such that

$$F(x, 0) = x_0 \quad \text{and} \quad F(x, 1) = f(x) \quad \forall x \in I^n$$

Since the constant map lifts trivially, using the homotopy Lifting theorem, we conclude that F lifts to an $\tilde{F}$. The map $\tilde{f} : I^n \rightarrow P$ defined by

$$\tilde{f}(x) = \tilde{F}(x, 1)$$

is the required lift. $\square$

3.2 Covering spaces

Definition 3.2.1. Let X be a Hausdorff space. A covering space for X consists of

1. a connected, Hausdorff space $\tilde{X}$
2. a continuous map $\mathcal{P} : \tilde{X} \rightarrow X$ of $\tilde{X}$ onto X
3. $\forall x \in X$, there exists a neighborhood V in X for which $\mathcal{P}^{-1}(V)$ is the disjoint union of a family of open sets S_α in $\tilde{X}$, each of which is mapped homeomorphically onto V by $\mathcal{P}$, i.e.

$$V = \coprod_{\alpha} S_{\alpha} \quad \text{and} \quad \mathcal{P}|_{S_{\alpha}} : S_{\alpha} \rightarrow V \text{ is a homeomorphism}$$

Any such V is said to be evenly covered by $\mathcal{P}$ and the open sets S_α in $\tilde{X}$ are called sheet over V .

Lemma 3.2.1. For each $x \in X$, the fiber above x is a discrete subspace of $\tilde{X}$ with a given cardinality, which does not depend on x . In particular, this means that all the fibers are homeomorphic.

Proof. Let V be an evenly covered neighborhood of x . We then have that each sheet S_α can only contain one point of $\mathcal{P}^{-1}(x)$, since otherwise the restriction of the projection to the corresponding sheet would not be injective, which contradicts the hypothesis that this restriction is a homeomorphism. Therefore, $\mathcal{P}^{-1}(x) \cap S_\alpha$ is a single point and since the S_α are open by hypothesis, this intersection is open in the subspace topology of $\mathcal{P}^{-1}(x)$. Since every point of $\mathcal{P}^{-1}(x)$ is open, $\mathcal{P}^{-1}(x)$ is discrete.

Since $\tilde{X}$ is connected, so is X . For every $\Omega \in \mathbb{N}$ less or equal that the cardinality of P , let H_Ω be the set of all $x \in X$ such that $\mathcal{P}^{-1}(x)$ has cardinality Ω . Let $x_0 \in H_\Omega$ and let V be an evenly covered neighborhood of x_0 in X . Every sheet over V contains precisely one element of the fiber over x_0 so the number of such sheets is also Ω . Therefore, for every $x \in V$, we have $x \in H_\Omega$. So that $x_0 \in V \subseteq H_\Omega$, which implies that H_Ω is open. Furthermore, we have that

$$H_\Omega^c = \bigcup_{n \neq \Omega} H_n \Rightarrow H_\Omega^c \text{ is open}$$

so that H_Ω is both open and closed and since X is connected, one of the H_Ω is equal X . $\square$

The following lemma states that covering spaces are actually locally trivial bundles with discrete fibers.

Lemma 3.2.2. Let $\mathcal{P} : \tilde{X} \rightarrow X$ be a covering space. Then $(\tilde{X}, X, \mathcal{P}, \mathcal{P}^{-1}(x))$ is a locally trivial bundle.

Theorem 3.2.1 (Unique Lifting Theorem). Let $\mathcal{P} : \tilde{X} \rightarrow X$ be a covering space, x_0 a point in X and $\tilde{x}_0$ a point in $\mathcal{P}^{-1}(x_0)$. Suppose that Z is a connected topological space and $f : Z \rightarrow X$ a continuous map which satisfies $f(z_0) = x_0$. If there is a lift $\tilde{f} : Z \rightarrow \tilde{X}$ of f to $\tilde{X}$ which satisfies $\tilde{f}(z_0) = \tilde{x}_0$, then this lift is unique.

Corollary 3.2.1. Let $\mathcal{P} : \tilde{X} \rightarrow X$ be a covering space, x_0 a point in X and $\tilde{x}_0$ a point in $\mathcal{P}^{-1}(x_0)$. Suppose $\alpha : [0, 1] \rightarrow X$ is a path in X with $\alpha(0) = x_0$. Then there exists a unique lift $\tilde{\alpha} : [0, 1] \rightarrow \tilde{X}$ of α to $\tilde{X}$ that satisfies $\tilde{\alpha}(0) = \tilde{x}_0$.

In the particular case of covering spaces, the homotopy lifting theorem implies a very important result, which is stated in the following corollary.

Corollary 3.2.2. Let $\mathcal{P} : \tilde{X} \rightarrow X$ be a covering space, x_0 a point in X and $\tilde{x}_0$ a point in $\mathcal{P}^{-1}(x_0)$ the fiber above x_0 . Let $\alpha, \beta : [0, 1] \rightarrow X$ be paths in X with $\alpha(0) = \beta(0) = x_0$ and suppose $F : [0, 1] \times [0, 1] \rightarrow X$ is a path homotopy from α to β , which means in particular that $\alpha(1) = \beta(1)$. If $\tilde{\alpha}, \tilde{\beta}$ are the unique lifts to P of α and β with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0$, then there exists a path homotopy $\tilde{F} : [0, 1] \times [0, 1] \rightarrow \tilde{X}$ from $\tilde{\alpha}$ to $\tilde{\beta}$ with $\mathcal{P} \circ \tilde{F} = F$. In particular, we have $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

Proof. The homotopy lifting theorem gives us the existence of a homotopy $\tilde{F} : [0, 1] \times [0, 1] \rightarrow \tilde{X}$ such that $\mathcal{P} \circ \tilde{F} = F$ and $\tilde{F}(s, 0) = \tilde{\alpha}(s)$. We need show that this homotopy is actually a path homotopy which ends at $\tilde{\beta}$.

First, we note that

$$\mathcal{P} \circ \tilde{F}(0, t) = F(0, t) = \tilde{x}_0 \quad \forall t \in [0, 1] \Rightarrow \gamma(t) \equiv \tilde{F}(0, t) \in \mathcal{P}^{-1}(x_0)$$

But for a covering space, the fibers are discrete and since γ is a continuous path, it has to be the constant path, i.e. $\gamma(t) = \tilde{x}_0$ for all $t \in [0, 1]$.

Furthermore, we have

$$\mathcal{P} \circ \tilde{F}(s, 1) = F(s, 1) = \beta \Rightarrow \rho(s) \equiv \tilde{F}(s, 1) \text{ is a path that lifts } \beta.$$

In addition, we have just shown that at least one endpoint is constant, so that $\rho(0) = \tilde{F}(0, 1) = \gamma(1) = \tilde{x}_0$. The unique lifting theorem then implies that $\rho = \tilde{\beta}$.

In the same way, we then have that $\tilde{F}(1, t)$ is a path in $\mathcal{P}^{-1}(y_0)$ which is discrete so the path is constant and $\tilde{F}(1, t) = \tilde{\alpha}(1) = \tilde{F}(1, 1) = \tilde{\beta}(1)$. $\square$

Covering spaces are especially useful because they give a way to compute higher homotopy groups.

Theorem 3.2.2. *Let $\mathcal{P} : \tilde{X} \rightarrow X$ be a covering space, x_0 a point in X and $\tilde{x}_0$ a point in $\mathcal{P}^{-1}(x_0)$. Then, for each $n \geq 2$, $\mathcal{P}_* : \pi_n(\tilde{X}, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$ is an isomorphism.*

3.3 Principal bundles

3.3.1 Definition and basic properties

Definition 3.3.1. *Let X be a differentiable manifold and G a Lie group. A smooth principal bundle over X with structure group G , or shorter a smooth G -bundle over X , is a triple $\mathcal{B} = (P, \mathcal{P}, \sigma)$, where P is a differentiable manifold, $\mathcal{P}$ is a smooth map of P onto X and $\sigma : P \times G \rightarrow P$, where we write $\sigma(p, g) = p \cdot g$, is a smooth right action of G on P such that the following conditions are satisfied:*

1. σ preserves the fibers of $\mathcal{P}$, i.e.

$$\mathcal{P}(p \cdot g) = \mathcal{P}(p) \quad \forall p \in P, g \in G \quad (3.1)$$

2. For each $x_0 \in X$, there exists an open neighborhood $V \ni x_0$ in X and a diffeomorphism $\Psi : \mathcal{P}^{-1}(V) \rightarrow V \times G$ of the form

$$\Psi(p) = (\mathcal{P}(p), \psi(p)) \quad (3.2)$$

where $\psi : \mathcal{P}^{-1}(V) \rightarrow G$ satisfies

$$\psi(p \cdot g) = \psi(p)g \quad \forall p \in \mathcal{P}^{-1}(V), g \in G$$

Example 3.3.1. See section 7.3.

Lemma 3.3.1. *Let $p \in P$. Then the fiber above $\mathcal{P}(p)$ coincides with the orbit of p under σ , i.e.*

$$\mathcal{P}^{-1}(\mathcal{P}(p)) = \{p \cdot g : g \in G\} = p \cdot G$$

In particular, this proves that all the fibers are diffeomorphic to the structure group G .

Proof. We prove the inclusion in both directions.

1. $\mathcal{P}^{-1}(\mathcal{P}(p)) \supseteq \{p \cdot g : g \in G\} = p \cdot G$: This a direct consequence of (3.1).

2. $\mathcal{P}^{-1}(\mathcal{P}(p)) \subseteq \{p \cdot g : g \in G\} = p \cdot G$: Let $p' \in \mathcal{P}^{-1}(\mathcal{P}(p'))$. We will show that there exists $g \in G$ such that $p' = p \cdot g$.

We first choose V and Ψ at $x_0 = \mathcal{P}(p) = \mathcal{P}(p')$. We then have:

$$\psi(p') = \psi(p)[(\psi(p))^{-1}(\psi(p'))] \equiv \psi(p)g$$

where $g \in G$ by definition.

$$\Rightarrow \psi(p') = \psi(p \cdot g) \Rightarrow \Psi(p \cdot g) = (\mathcal{P}(p \cdot g), \psi(p \cdot g)) = (\mathcal{P}(p'), \psi(p')) = \Psi(p')$$

Since Ψ is a diffeomorphism on V , we have:

$$p' = p \cdot g$$

□

Lemma 3.3.2. *$(P, X, \mathcal{P}, G)$ is a locally trivial bundle with local trivializations (V, Ψ^{-1}) . In particular, this implies that P is Hausdorff.*

Proof. The only thing one needs to verify is that Ψ^{-1} has the “projection property”. On V , Ψ^{-1} is a diffeomorphism so:

$$\Psi(p) = (\mathcal{P}(p), \psi(p)) \Leftrightarrow \Psi^{-1}(\mathcal{P}(p), \psi(p)) = p$$

So that

$$\mathcal{P} \circ \Psi^{-1}(\mathcal{P}(p), \psi(p)) = \mathcal{P}(p)$$

The previous lemma tells us that ψ restricted to a fiber is onto, i.e. $\forall g \in G$, there exists p' in the fiber such that $\psi(p') = g$. Furthermore, we also that there exists $g' \in G$ such that $\psi(p') = \psi(p \cdot g')$. We therefore finally obtain

$$\mathcal{P} \circ \Psi^{-1}(\mathcal{P}(p), \psi(p)) = \mathcal{P} \circ \Psi^{-1}(\mathcal{P}(p), \psi(p')) = \mathcal{P} \circ \Psi^{-1}(\mathcal{P}(p), g) = \mathcal{P}(p)$$

Since $\mathcal{P}$ is onto and the previous equation holds for any g , we have shown that

$$\mathcal{P} \circ \Psi^{-1}(x, g) = x \quad \forall x \in V, \forall g \in G$$

□

Example 3.3.2. *The simplest example of a principal bundle is the trivial bundle $(X \times G, X, \mathcal{P}, G)$, where $\mathcal{P} : X \times G \rightarrow X$ is simply the projection on the first factor and the action of G on $X \times G$ is defined by $(x, h) \cdot g = (x, hg)$. One can then take $V = X$ and Ψ to be the identity on $\mathcal{P}^{-1}(V) = X \times G$.*

Lemma 3.3.3. *Let $\mathcal{B}' = (P', \mathcal{P}', \sigma')$ be a principal G -bundle over X' and suppose that X is a topological subspace of X' . We define*

$$P = (\mathcal{P}')^{-1}(X), \quad \mathcal{P} = \mathcal{P}'|_P \quad \text{and} \quad \sigma = \sigma'|_{P \times G}$$

Then (V, Ψ) where $V = V' \cap X$, $\Psi = \Psi'|_{\mathcal{P}^{-1}(V)}$ and (V', Ψ') is any trivialization of $\mathcal{B}'$ are trivializations of the principal G -bundle over X $\mathcal{B}'|_X = (P, \mathcal{P}, \sigma)$ called the restriction of $\mathcal{B}'$ to X .

Definition 3.3.2. *Let $\{V_\alpha, \Psi_\alpha\}_{\alpha \in \mathcal{A}}$ a trivializing cover. Now take $\alpha, \beta \in \mathcal{A}$ and $V_\alpha \cap V_\beta \neq \emptyset$. We then define the map $g_{\alpha\beta} : V_\alpha \cap V_\beta \rightarrow G$ by*

$$g_{\alpha\beta}(x) = \psi_\alpha(p)(\psi_\beta(p))^{-1}$$

where $x \in V_\alpha \cap V_\beta$ and $p \in \mathcal{P}^{-1}(x)$ is any element of the fiber above x . These maps are called the transition functions of the principal bundle associated with the trivializing cover $\{V_\alpha, \Psi_\alpha\}_{\alpha \in \mathcal{A}}$.

Lemma 3.3.4. *This is well-defined, i.e. $\psi_\alpha(p)(\psi_\beta(p))^{-1}$ is constant on fibers. Furthermore, the $g_{\alpha\beta}$ are smooth.*

Proof. Now let $p' \neq p$ be any point in fiber. We then have by Lemma 3.3.1 that there exists $g \in G$ such that $p' = p \cdot g$. We therefore get

$$\psi_\alpha(p')(\psi_\beta(p'))^{-1} = \psi_\alpha(p)g(\psi_\beta(p)g)^{-1} = \psi_\alpha(p)(\psi_\beta(p))^{-1}$$

The $g_{\alpha\beta}$ are smooth because the ψ_γ are smooth and G is a Lie group. $\square$

Lemma 3.3.5. *For each $x \in V_\alpha \cap V_\beta$,*

$$(\psi_\alpha|_{\mathcal{P}^{-1}(x)}) \circ (\psi_\beta|_{\mathcal{P}^{-1}(x)})^{-1}(g) = g_{\alpha\beta}(x)g$$

for all $g \in G$.

Proof. For every $x \in V_\alpha \cap V_\beta$, ψ_α and ψ_β are diffeomorphisms of the fiber above x onto G so that

$$(\psi_\alpha|_{\mathcal{P}^{-1}(x)}) \circ (\psi_\beta|_{\mathcal{P}^{-1}(x)})^{-1} : G \rightarrow G$$

is also a diffeomorphism.

Let $(\psi_\beta|_{\mathcal{P}^{-1}(x)})^{-1}(g) = p$. We then have:

$$g = \psi_\beta(p) \quad \text{and} \quad (\psi_\alpha|_{\mathcal{P}^{-1}(x)}) \circ (\psi_\beta|_{\mathcal{P}^{-1}(x)})^{-1}(g) = \psi_\alpha(p)$$

Since $p \in \mathcal{P}^{-1}(x)$, we obtain

$$g_{\alpha\beta}(x)g = \psi_\alpha(p)$$

which proves the result. $\square$

Lemma 3.3.6. *If $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$ and $x \in V_\alpha \cap V_\beta \cap V_\gamma$, then the transition functions satisfy the following properties:*

1. *The cocycle condition: $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$*
2. *$g_{\alpha\alpha}(x) = e$*
3. *$g_{\beta\alpha}(x) = (g_{\alpha\beta}(x))^{-1}$*

Proof. The properties are direct consequences of the definition of the transition functions. $\square$

3.3.2 Bundle maps and equivalence

Definition 3.3.3. *Let $\mathcal{B}_1(\mathcal{P}_1 : P_1 \rightarrow X_1)$ and $\mathcal{B}_1(\mathcal{P}_2 : P_2 \rightarrow X_2)$ be two G -bundles. Since there is no risk for confusion, we write the two actions with the same dot $\cdot$. A principal bundle map from $\mathcal{B}_1$ to $\mathcal{B}_2$ is a smooth map $\tilde{f} : P_1 \rightarrow P_2$ such that*

$$\tilde{f}(p \cdot g) = \tilde{f}(p) \cdot g \quad \forall g \in G, \forall p \in P_1$$

Lemma 3.3.7. *A principal bundle map $\tilde{f}$ carries each fiber $\mathcal{P}_1$ diffeomorphically onto a fiber of $\mathcal{P}_2$*

Definition 3.3.4. *Let $\mathcal{B}_1(\mathcal{P}_1 : P_1 \rightarrow X_1)$, $\mathcal{B}_1(\mathcal{P}_2 : P_2 \rightarrow X_2)$ be two G -bundles and $\tilde{f} : P_1 \rightarrow P_2$ a principal bundle map. Then $\tilde{f}$ induces or covers a map $f : X_1 \rightarrow X_2$, called the induced map defined by*

$$\mathcal{P}_2 \circ \tilde{f} = f \circ \mathcal{P}_1$$

Lemma 3.3.8. *The induced map f is necessarily smooth. Furthermore, if $\tilde{f}$ induces a diffeomorphism $f : X_1 \rightarrow X_2$, then $\tilde{f} : P_1 \rightarrow P_2$ is also a diffeomorphism and $\tilde{f}^{-1} : P_2 \rightarrow P_1$ is a bundle map of $\mathcal{B}_2$ onto $\mathcal{B}_1$.*

Definition 3.3.5. *We now define equivalence of bundles.*

1. Let $\mathcal{B}_1(\mathcal{P}_1 : P_1 \rightarrow X)$, $\mathcal{B}_1(\mathcal{P}_2 : P_2 \rightarrow X)$ be two G -bundles over the same base space X . Then a bundle map $\tilde{f} : P_1 \rightarrow P_2$ is called an equivalence, in which case $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be equivalent, if the induced map $f : X \rightarrow X$ is the identity id_X .
2. If $\mathcal{B}(\mathcal{P} : P \rightarrow X)$ is a fixed G -bundle over X , then an equivalence $\tilde{f} : P \rightarrow P$ is called an automorphism of the bundle.
3. A G -bundle over X is said to be trivial if it is equivalent to the trivial G -bundle $\mathcal{P} : X \times G \rightarrow X$ over X defined previously.

Lemma 3.3.9. *The equivalence defined in the previous definition defines an equivalence relation.*

Proof. Reflexivity is trivial, since the identity is an equivalence. Symmetry is a consequence of Lemma 3.3.8. We now show transitivity. Let $\mathcal{B}_1(\mathcal{P}_1 : P_1 \rightarrow X)$, $\mathcal{B}_2(\mathcal{P}_2 : P_2 \rightarrow X)$, $\mathcal{B}_3(\mathcal{P}_3 : P_3 \rightarrow X)$ be 3 G -bundles over X and $\tilde{f} : P_1 \rightarrow P_2$ and $\tilde{g} : P_2 \rightarrow P_3$ be equivalences, i.e. $\mathcal{P}_2 \circ \tilde{f} = \mathcal{P}_1$ and $\mathcal{P}_3 \circ \tilde{g} = \mathcal{P}_2$. Then $\tilde{h} \equiv \tilde{g} \circ \tilde{f}$ is clearly smooth and

$$\tilde{h}(p_1 \cdot g) = \tilde{h}(p_1) \cdot g$$

so $\tilde{h}$ is a principal bundle map. Furthermore,

$$\mathcal{P}_3 \circ \tilde{h} = \mathcal{P}_2 \circ \tilde{f} = \mathcal{P}_1$$

so $\tilde{h}$ is an equivalence. □

Lemma 3.3.10. *A principal G -bundle $\mathcal{B}$ is trivial if and only if it has a global trivialization, i.e. if and only if one can take $V = X$ as a trivializing neighborhood.*

Proof. 1. $\Rightarrow$: The bundle is trivial so there exists $\tilde{f} : P \rightarrow X \times G$ such that $\mathcal{P}_X \circ \tilde{f} = \mathcal{P}$. We take $V = X$ and $\psi = \tilde{f}$, then

$$\psi(p) = \tilde{f}(p) = (\mathcal{P}(p), \mathcal{P}_Y \circ \tilde{f}(p))$$

$$\Rightarrow \psi(p \cdot g) = \tilde{f}(p) \cdot g = (\mathcal{P}(p), \mathcal{P}_Y \circ \tilde{f}(p)g) \Rightarrow \mathcal{P} \circ \tilde{f}(p \cdot g) = \mathcal{P}_Y \circ \tilde{f}(p)g$$

and $\tilde{f}$ is a homeomorphism since it induces the identity.

2. $\Leftarrow$: Suppose $V = X$ and $\psi : P \rightarrow X \times G$ the trivialization, then we define $\tilde{f}(p) = \psi(p)$ and we have $\mathcal{P}_X \circ \psi = \mathcal{P}$ by definition. □

On a principal there always exists a local cross-sections on the trivializing neighborhoods.

Definition 3.3.6. *Let $\mathcal{B}$ be a G -bundle over some space X and $\Psi : \mathcal{P}^{-1}(V) \rightarrow V \times G$ a trivialization on V of this bundle. Then the canonical cross-section $s_V : V \rightarrow P$ associated with that trivialization is defined as*

$$s_V(x) = \Psi^{-1}(x, e) \quad \forall x \in V$$

Example 3.3.3. *See Hopf Bundle in section 7*

Proposition 3.3.1. *Let $\mathcal{B}$ be a principal G -bundle. Then there exists a one-to-one correspondence between smooth local cross sections and local trivializations of this bundle.*

Proof. We have just seen that one can associate a local cross-section with any trivialization.

We now proceed to show that one can always associate a trivialization to a smooth local cross-section. Suppose we are given a smooth local cross-section $s : V \rightarrow \mathcal{P}^{-1}(V)$ on some open set V of the base space. We have:

$$\mathcal{P}^{-1}(V) = \bigcup_{x \in V} \mathcal{P}(x) = \bigcup_{x \in V} s(x) \cdot G$$

since $s(x) \in \mathcal{P}^{-1}(x)$. We can therefore define a map $\Psi : \mathcal{P}^{-1}(V) \rightarrow V \times G$ by

$$\Psi(s(x) \cdot g) = (x, g)$$

Ψ is trivially a bijection. Furthermore, since $\mathcal{P}(s(x) \cdot g) = x$, we can write Ψ as

$$\Psi(s(x) \cdot g) = (\mathcal{P}(s(x) \cdot g), \psi(s(x) \cdot g))$$

where we have defined $\psi(s(x) \cdot g) = g$. We therefore have:

$$\psi((s(x) \cdot g) \cdot g') = gg' = \psi(s(x) \cdot g)g' \Rightarrow \psi(p \cdot g') = \psi(p)g' \quad \forall p \in \mathcal{P}^{-1}(V), \forall g' \in G$$

We now prove the smoothness of Ψ and Ψ^{-1} . Since Ψ is a bijection, we have

$$\Psi^{-1}(x, g) = s(x) \cdot g$$

which is smooth by composition of smooth maps. We start by proving the continuity of ψ . To do so, we fix $p = s(x) \cdot g$ and we choose a trivialization (V', Ψ') at $\mathcal{P}(p)$ with $\Psi' = (\mathcal{P}, \psi')$. We then have:

$$\psi'(p) = \psi'(s(\mathcal{P}(p)) \cdot g) = (\psi'(s(\mathcal{P}(p))))g \Rightarrow \psi(p) = g = \psi'(p)(\psi'(s(\mathcal{P}(p))))^{-1}$$

The continuity of the first component of Ψ is evident. We have therefore proven that Ψ is a diffeomorphism. $\square$

Theorem 3.3.1. *A principal G -bundle $\mathcal{P} : P \rightarrow X$ is trivial if and only if it admits a global cross-section $s : X \rightarrow P$.*

Proof. This is a direct consequence of Lemma 3.3.10. $\square$

Lemma 3.3.11. *Let (Ψ_i, V_i) and (Ψ_j, V_j) be trivializations and $s_{V_i} \equiv s_i$ and $s_{V_j} \equiv s_j$ be the corresponding canonical cross-sections. For every $x \in V_i \cap V_j$, we have*

$$s_i(x) = s_j(x) \cdot g_{ji}(x)$$

where g_{ji} is the corresponding transition function.

Proof. Follows from a simple computation. $\square$

Corollary 3.3.1. *Let G be a topological group and n a positive integer. Then any principal G -bundle over the n -cube I^n is trivial.*

Proof. Let $\mathcal{P} : P \rightarrow I^n$ be a principal G -bundle over I^n . Since I^n is convex and therefore contractible, the identity map on I^n is nullhomotopic. Let F be the corresponding homotopy, i.e. $F(x, 0) = x_0$ and $F(x, 1) = x$ for all $x \in I^n$. Let $f : I^n \rightarrow I^n$ be the constant map $f(x) = x_0$ and $p_0 \in \mathcal{P}^{-1}(x_0)$. Then the constant map $\tilde{f} : I^n \rightarrow P$, $\tilde{f}(x) = p_0$ lifts f . Since $F(x, 0) = f(x)$, the Homotopy Lifting Theorem 3.1.1 implies that there exists a homotopy $\tilde{F}$ such that $\mathcal{P} \circ \tilde{F} = F$, so that in particular we have

$$\mathcal{P} \circ \tilde{F}(x, 1) = x = \text{id}(x)$$

We can therefore define a global cross-section $s(x) = \tilde{F}(x, 1)$ on our principal bundle, which then implies that the principal bundle is trivial. $\square$

Theorem 3.3.2. *Any principal G -bundle over a contractible, paracompact space is trivial.*

Remark 3.3.1. *It is simple to convince that any refinement of a trivializing cover also is a trivializing cover. Therefore, if one considers two principal G -bundles over the same base space X , one can always assume without loss of generality (by considering the trivializing cover consisting of the intersection of the respective trivializing covers) that they have the two principal bundles have the same trivializing covers.*

We can formulate in a new way of testing the equivalence of principal bundles.

Lemma 3.3.12. Let $\mathcal{B}_1(\mathcal{P}_1 : P_1 \rightarrow X)$, $\mathcal{B}_2(\mathcal{P}_2 : P_2 \rightarrow X)$ be two G -bundles over the same base space X and suppose that $\{V_j\}_{j \in J}$ is a trivializing cover for $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $g_{ij}^1, g_{ij}^2 : V_i \cap V_j \rightarrow G$ be the corresponding transition functions for $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent if and only if there exists a family of continuous map $\lambda_l : V_l \rightarrow G$, $l \in J$ such that

$$g_{ji}^2(x) = (\lambda_j(x))^{-1} g_{ji}^1(x) \lambda_i(x)$$

for all $x \in V_i \cap V_j$.

Theorem 3.3.3. Let X be a smooth manifold, G a Lie group and $\{V_j\}_{j \in J}$ an open cover of X . Suppose that for each $i, j \in J$ such that $V_i \cap V_j \neq \emptyset$, we have a smooth map $g_{ji} : V_i \cap V_j \rightarrow G$ and that these map have the property that if $V_i \cap V_j \cap V_k \neq \emptyset$, then

$$g_{kj}(x) g_{ji}(x) = g_{ki}(x)$$

for all $x \in V_i \cap V_j \cap V_k$. Then there exists a principal G -bundle over X , unique up to equivalence, which has the V_j where $j \in J$ as trivializing neighborhoods and the g_{ji} as transition functions.

Proof. Proved in Naber page 223-226. □

3.3.3 Fundamental vector field

Definition 3.3.7. Let G be a matrix Lie group, P a manifold and $\sigma : P \times G \rightarrow P$ a smooth right action of G on P . We define the map $\sigma_p : G \rightarrow P$ by

$$\sigma_p(g) = p \cdot g$$

for every $p \in P$. We furthermore define the map $\sigma : \mathcal{G} \rightarrow \mathcal{X}(P)$ where $\mathcal{G}$ is identified with a set of matrices by

$$\sigma(A)(p) = (\sigma_p)_{*id}(A)$$

where $A \in \mathcal{G}$.

Theorem 3.3.4. Let G be a matrix Lie group, P a manifold and $\sigma : P \times G \rightarrow P$ a smooth right action of G on P . Then for every $A \in \mathcal{G}$, $\sigma(A)$ is indeed a smooth vector field, i.e. $\sigma(A) \in \mathcal{X}(P)$ and can be rewritten as

$$\sigma(A)(p) = \frac{d}{dt} (p \cdot \exp(tA))|_{t=0}$$

Furthermore, $\sigma(A)$ is complete and its 1-parameter group of diffeomorphisms $\{(\sigma(A))_t\}_{t \in \mathbb{R}}$ is given by

$$(\sigma(A))_t(p) = p \cdot \exp(tA)$$

The map $A \rightarrow \sigma(A)$ of $\mathcal{G}$ to $\mathcal{X}$ is linear and satisfies

$$[\sigma(A), \sigma(B)] = \sigma([A, B])$$

for all $A, B \in \mathcal{G}$.

If the action is effective and $A \neq 0$, then $\sigma(A) \neq 0$ on P and if the action is free and $A \neq 0$, the $\sigma(A)$ is never zero.

Proof. We define $\alpha : \mathbb{R} \rightarrow G$ by

$$\alpha(t) = \exp(tA)$$

Then

$$\alpha'(0) = A \Rightarrow \sigma(A)(p) = (\sigma_p)_{*id}(A) = (\sigma_p)_{*id}(\alpha'(0)) = (\sigma_p \circ \alpha)'(0)$$

By definition, we have

$$(\sigma_p \circ \alpha)(t) = p \cdot \exp(tA)$$

which proves the smoothness of $\sigma(A)$ as well as its alternative formula. We have

$$\begin{aligned} \sigma(A)(\alpha_p(t_0)) &= (\sigma_{\sigma(p, \exp(t_0 A))} \circ \alpha)'(0) \\ &= (\sigma(p, \exp(A(t + t_0))))'(0) \\ &= 1 \cdot (\sigma(p, \alpha_p(t)))'(t_0) = \alpha'_p(t_0) \end{aligned}$$

$\sigma(A)$ is linear by the properties of the pushforward.

Let $A, B \in \mathcal{G}$ and $\mathbf{A}, \mathbf{B}$ the corresponding left-invariant vector fields. We then have:

$$[A, B] = [\mathbf{A}, \mathbf{B}]_{\text{id}} = \lim_{t \rightarrow 0} \frac{1}{t} (ad_{\exp(tA)}(B) - B)$$

where we used Lemma 2.1.5 and skipped some simple calculations involving the left-invariance of $\mathbf{A}$ and $\mathbf{B}$.

We denote the 1-parameter group of $\sigma(A)$ by $\{\xi_t\}_{t \in \mathbb{R}}$ where $\xi_t(p) = p \cdot \exp(tA)$. We then have

$$[\sigma(A), \sigma(B)] = \lim_{t \rightarrow 0} \frac{1}{t} [(\xi_{-t} \circ \sigma_{p \cdot \exp(tA)})_{* \text{id}}(B) - (\sigma_p)_{* \text{id}}(B)]$$

where we again use Lemma 2.1.5. We furthermore have

$$\xi_{-t} \circ \sigma_{p \cdot \exp(tA)} = \sigma \circ Ad_{\exp(tA)}(g)$$

by some simple computations and using the definitions, so that

$$\xi_{-t} \circ \sigma_{p \cdot \exp(tA)} = \sigma_p \circ Ad_{\exp(tA)} \Rightarrow (\xi_{-t} \circ \sigma_{p \cdot \exp(tA)})_{* \text{id}} = (\sigma_p)_{* \text{id}} \circ ad_{\exp(tA)}$$

So that finally, we obtain

$$[\sigma(A), \sigma(B)]_p = \lim_{t \rightarrow 0} \frac{1}{t} ((\sigma_p)_{* \text{id}} \circ ad_{\exp(tA)}(B) - (\sigma_p)_{* \text{id}}(B)) \quad (3.3)$$

$$= (\sigma_p)_{* \text{id}} \left(\lim_{t \rightarrow 0} \frac{1}{t} (ad_{\exp(tA)}(B) - B) \right) \quad (3.4)$$

$$= (\sigma_p)_{* \text{id}}([A, B]) = \sigma([A, B])(p) \quad (3.5)$$

Finally, let us assume that the action is effective and that $A \neq 0$. By contradiction, let us furthermore assume that $\sigma(A)(p) = 0$ for all $p \in P$. Then every integral curve of $\sigma(A)$ is constant so that

$$p \cdot \exp(tA) = p \quad \forall t \in \mathbb{R} \Rightarrow \exp(tA) = \text{id} \quad \forall t \in \mathbb{R}$$

where the implication comes from the effectiveness of the action. But $\exp$ is local diffeomorphism near 0 of $\mathcal{G}$ which contradicts the assumption that $A \neq 0$. $\square$

Definition 3.3.8. Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X , where G is a matrix Lie group. The fundamental vector field on P determined by A is the vector field $\sigma(A) \in \mathcal{X}(P)$ defined above, which in this context we denote by $A^\#$.

Definition 3.3.9. Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X , where G is a matrix Lie group. For every $p \in P$, the tangent space $T_p(P)$ contains a subspace $\text{Vert}_p(P)$ (consisting of tangent vectors at p to smooth curves in the fiber containing p) that is isomorphic to $T_p(G)$ and therefore to $\mathcal{G}$ (since the fibers are submanifolds of P which are diffeomorphic to G). We call $\text{Vert}_p(P)$ the vertical subspace of $T_p(P)$ and refer to its elements as vertical vectors at p .

Lemma 3.3.13. Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X , where G is a matrix Lie group. For each $p \in P$ the mapping $A \rightarrow A^\#(p)$ is an isomorphism of the Lie algebra $\mathcal{G}$ of G onto the vertical subspace $\text{Vert}_p(P)$.

Proof. For every $A \in \mathcal{G}$, we have

$$A^\#(p) = \frac{d}{dt} (p \cdot \exp(tA))|_{t=0}$$

But since $\mathcal{P}(p \cdot \exp(tA)) = \mathcal{P}(p)$, we have that $\alpha(t) = p \cdot \exp(tA)$ is contained in the fiber containing p , which implies that $A^\#(p)$ is a vertical vector at p for every $A \in \mathcal{G}$.

Linearity of the map is trivial. Since the action of a principal bundle is free, we have that $A^\#(p)$ is never zero except if $A = 0$, so that the kernel of the map is trivial, which in turn implies that the map $A \rightarrow A^\#(p)$ is one-to-one. Finally, since $\dim \mathcal{G} = \dim \text{Vert}_p(P)$, the map is in fact an isomorphism. $\square$

For every $g \in G$, σ_g is a diffeomorphism on P , so that for any $A \in \mathcal{G}$, we have that $(\sigma_g)_*(A^\#)$ is a smooth vector field. The following lemma tells us that it is much more.

Lemma 3.3.14. *Let $g \in G$ and $A \in \mathcal{G}$. We then have that*

$$(\sigma_g)_*(A^\#) = (ad_{g^{-1}}(A))^\#$$

Proof. Since $\sigma_g(p \cdot g^{-1}) = p$, we must only prove that

$$(\sigma_g)_{*p \cdot g^{-1}}(A^\#(p \cdot g^{-1})) = (ad_{g^{-1}}(A))^\#(p)$$

The rest follows from a simple computation. $\square$

4 Connections and gauge equivalence

4.1 Connections on principal bundles

Definition 4.1.1. *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle, where we assume that G is a matrix Lie group (and $\mathcal{G}$ its Lie algebra). A connection or gauge field on $\mathcal{B}$ (or on P) is a smooth $\mathcal{G}$ -valued 1-form ω on P which satisfies the following properties:*

1. $(\sigma_g)^*\omega = ad_{g^{-1}} \circ \omega$ for all $g \in G$, i.e. for all $g \in G$, $p \in P$ and $v \in T_{p \cdot g^{-1}}(P)$ we have

$$\omega_p((\sigma_g)_{*p \cdot g^{-1}}(v)) = g^{-1}\omega_{p \cdot g^{-1}}(v)g$$

2. $\omega(A^\#) = A$ for all $A \in \mathcal{G}$, i.e. for all $A \in \mathcal{G}$ and $p \in P$, we have

$$\omega_p(A^\#(p)) = A$$

Example 4.1.1. *See Hopf bundle at section 8.*

Definition 4.1.2. 1. *In the context of smooth principal bundles with connections, a local cross-section $s : V \rightarrow \mathcal{P}^{-1}(V)$ of $\mathcal{B}$ is also known as local gauge (especially in the physics literature)*

2. *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a principal G -bundle with a connection ω , then the pullback $\mathcal{A} = s^*\omega$ of ω to $V \subseteq X$ is called a local gauge potential in the gauge s on X*

3. *Choosing a different local gauge, or equivalently (see Proposition 3.3.1) choosing a different trivialization of the bundle, is called a local gauge transformation.*

Although, as one might imagine, local gauge transformations can yield very different local gauge potentials, the following lemma shows that local gauge transformations actually have a relatively simple generic form.

Lemma 4.1.1. *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle with a connection ω . Let (V_i, Ψ_i) and (V_j, Ψ_j) be trivializations of the bundle with $V_i \cap V_j \neq \emptyset$, $g_{ij} : V_i \cap V_j \rightarrow G$ the corresponding transition function, $s_i, s_j : V_k \rightarrow P$ the associated canonical cross-sections and Θ the Cartan canonical 1-form of G . Writing $\mathcal{A}_k \equiv s_k^*\omega$ and $\Theta_{ij} \equiv g_{ij}^*\Theta$, we have the following relation between the two different local gauge potentials:*

$$\mathcal{A}_j = ad_{g_{ij}^{-1}} \circ \mathcal{A}_i + \Theta_{ij}$$

or more precisely, for every $x \in V_i \cap V_j$ and every $v \in T_x(X)$, we have

$$\mathcal{A}_j(x)(v) = ad_{g_{ij}(x)^{-1}}(\mathcal{A}_i(x)(v)) + \Theta_{ij}(x)(v) = g_{ij}(x)^{-1}(\mathcal{A}_i(x)(v))g_{ij}(x) + \Theta_{g_{ij}(x)}((g_{ij})_{*x}(v))$$

Proof. We fix $x \in V_i \cap V_j$ and $v \in T_x(X)$.

We have already shown that $s_j(x) = s_i(x) \cdot g_{ij}(x)$. We use another standard result from differential geometry by writing:

$$\sigma_{*(p_0, g_0)}(w) = (\sigma_1)_{*p_0}(w_P) + (\sigma_2)_{*g_0}(w_G)$$

where $w = (w_P, w_G) \in T_{(p_0, g_0)}(P \times G)$ and $\sigma_1 : P \rightarrow P$ and $\sigma_2 : G \rightarrow P$ defined by $\sigma_1(p) = p \cdot g_0$ and $\sigma_2(g) = p_0 \cdot g$. We therefore have

$$\begin{aligned} (s_j)_{*x}(v) &= (\sigma \circ (s_i, g_{ij}))_{*x}(v) \\ &= \sigma_{*(s_i(x), g_{ij}(x))} \circ ((s_i)_{*x}(v), (g_{ij})_{*x}(v)) \\ &= (\sigma_1 \circ s_i)_{*x}(v) + (\sigma_2 \circ g_{ij})_{*x}(v) \\ \Rightarrow \mathcal{A}_j(x)(v) &= \omega_{s_j(x)}((s_j)_{*x}(v)) \\ &= \omega_{s_j(x)}((\sigma_1 \circ s_i)_{*x}(v)) + \omega_{s_j(x)}((\sigma_2 \circ g_{ij})_{*x}(v)) \end{aligned}$$

We now have

$$\sigma_1 \circ s_i(y) = \sigma_{g_{ij}(x)} \circ s_i(y) \Rightarrow (\sigma_1 \circ s_i)_{*x}(v) = (\sigma_{g_{ij}(x)})_{*s_i(x)}((s_i)_{*x}(v))$$

We therefore obtain

$$\begin{aligned} \omega_{s_j(x)}((\sigma_1 \circ s_i)_{*x}(v)) &= \omega_{s_j(x)}((\sigma_{g_{ij}(x)})_{*s_i(x)}((s_i)_{*x}(v))) \\ &= \omega_{s_j(x)}((\sigma_{g_{ij}(x)})_{*s_j(x) \cdot g_{ij}(x)^{-1}}((s_i)_{*x}(v))) \\ &= ad_{g_{ij}(x)^{-1}}(\omega_{s_j(x) \cdot g_{ij}(x)^{-1}}((s_i)_{*x}(v))) \\ &= ad_{g_{ij}(x)^{-1}}(\mathcal{A}_i(x)(v)) \end{aligned}$$

where in the third line we used the first property of a connection.

Let $A \in \mathcal{G}$ be the unique element of the Lie algebra $\mathcal{G}$ whose left-invariant vector field $\mathbf{A}$ satisfies $\mathbf{A}(g_{ij}(x)) = (g_{ij})_{*x}(v)$. We then get

$$\Theta_{ij}(x)(v) = \Theta_{g_{ij}(x)}(\mathbf{A}(g_{ij}(x))) = A$$

By definition, we have $A^\#(s_i(x) \cdot g_{ij}(x)) = (\sigma_{s_i(x) \cdot g_{ij}(x)})_{*e}(A)$ and also $\sigma_{s_i(x) \cdot g_{ij}(x)}(g) = \sigma_2 \circ L_{g_{ij}(x)}(g)$. This implies

$$\begin{aligned} (\sigma_{s_i(x) \cdot g_{ij}(x)})_{*e}(A) &= (\sigma)_{*g_{ij}(x)}((L_{g_{ij}(x)})_{*e}(A)) \\ &= (\sigma)_{*g_{ij}(x)}(\mathbf{A}(g_{ij}(x))) \\ &= (\sigma_2 \circ g_{ij})_{*x}(v) \\ \Rightarrow A^\#(s_i(x) \cdot g_{ij}(x)) &= (\sigma_2 \circ g_{ij})_{*x}(v) \end{aligned}$$

So that finally, we have

$$\begin{aligned} \omega_{s_j(x)}((\sigma_2 \circ g_{ij})_{*x}(v)) &= \omega_{s_j(x)}(A^\#(s_i(x) \cdot g_{ij}(x))) \\ &= \omega_{s_j(x)}(A^\#(s_j(x))) \\ &= A \end{aligned}$$

Combining these results, we have obtained the formula

$$\mathcal{A}_j(x)(v) = ad_{g_{ij}(x)^{-1}}(\mathcal{A}_i(x)(v)) + \Theta_{ij}(x)(v)$$

□

The above transformation law can be rewritten in a more suggestive way, which also generally the way it is encountered in the physics literature.

Lemma 4.1.2. *The local gauge transformation presented in the previous lemma can be rewritten as*

$$\mathcal{A}_j = g_{ij}^{-1} \mathcal{A}_i g_{ij} + g_{ij}^{-1} dg_{ij}$$

Proof. Let α be a curve in X such that $\alpha'(0) = v$. Then by unwinding the definitions, one obtains:

$$\begin{aligned}\Theta_{g_{ij}(x)}((g_{ij})_*x(v)) &= \Theta_{g_{ij}(x)}((g_{ij})_*x(\alpha'(0))) \\ &= (L_{g_{ij}(x)} \circ g_{ij} \circ \alpha)'(0) \\ &= (g_{ij}(x))^{-1}(g_{ij} \circ \alpha)'(0) \\ &= (g_{ij}(x))^{-1}dg_{ij}(x)(v)\end{aligned}$$

where in the last line we used the standard coordinates in the matrix group G . Combining this result with Lemma 2.2.17 yields the sought formula. $\square$

Since one often works in local coordinate patches (especially in physics), the following lemma states the form of the local gauge transformations in local coordinates.

Lemma 4.1.3. *Let (U, ϕ) with $U \subseteq V_i \cap V_j$ be a chart of X . The local gauge transformation of Lemma 4.1.1 then becomes:*

$$(s_j \circ \phi^{-1})^*\omega = ad_{g_{ij} \circ \phi^{-1}} \circ (s_i \circ \phi^{-1})^*\omega + (g_{ij} \circ \phi^{-1})^*\omega$$

The remarkable thing is one can invert the process of local gauge transformations in the sense that a family of local gauge potentials whose domains cover X and that are consistent on the overlaps piece together to give a connection form on the bundle.

Theorem 4.1.1. *Let G be a matrix Lie group with Lie algebra $\mathcal{G}$ and $\mathcal{B} = (P, \mathcal{P}, \sigma)$ a smooth principal G -bundle over X . Let $\{V_i, \Psi_i\}_{i \in J}$ be a trivializing cover. Suppose that for each $j \in J$, $\mathcal{A}_j$ is a $\mathcal{G}$ -valued 1-form on V_j and that, whenever $V_j \cap V_i \neq \emptyset$,*

$$\mathcal{A}_j = ad_{g_{ij}^{-1}} \circ \mathcal{A}_i + \Theta_{ij} \quad \text{on } V_i \cap V_j$$

where $g_{ij} : V_i \cap V_j \rightarrow G$ is the transition function and $\Theta_{ij} = g_{ij}^*\Theta$ is the pullback by g_{ij} of the Cartan 1-form Θ for G . Then there exists a unique connection form ω on P such that for each $j \in J$, $\mathcal{A}_j = s_j^*\omega$ where $s_j : V_j \rightarrow \mathcal{P}^{-1}(V_j)$ is the canonical cross-section associated with the trivialization (V_j, Ψ_j) .

Proof. Let (V, Ψ) be any trivialization for $\mathcal{B}$ and $s : V \rightarrow \mathcal{P}^{-1}(V)$ the corresponding canonical cross-section. Let $(x_0, g_0) \in V \times G$ and $(v_1, v_2) \in T_{(x_0, g_0)}(V \times G)$. We have $(\sigma)_{*(x_0, g_0)}(v_1, v_2) = (\sigma_1)_{*x_0}(v_1) + (\sigma_2)_{*g_0}(v_2)$ where $\sigma_1(x) = x \cdot g_0$ and $\sigma_2(g) = x_0 \cdot g$. Furthermore, we have that

$$\Psi^{-1}(x_0, g_0) = s(x_0) \cdot g_0$$

Therefore, we get

$$\begin{aligned}(\Psi)_{*(x_0, g_0)}(v_1, v_2) &= (\sigma(s(\cdot), \cdot))_{*(x_0, g_0)}(v_1, v_2) \\ &= (\sigma_1)_{*s(x_0)}(s_{*x_0}(v_1)) + (\sigma(s(x_0), \cdot))_{*g_0}(v_2)\end{aligned}$$

Furthermore, we have

$$\begin{aligned}(\sigma(s(x_0), \cdot))_{*g_0}(v_2) &= (\sigma(s(x_0), \cdot))_{*g_0}((L_{g_0})_{*e}(A)) \\ &= \frac{d}{dt}\big|_0 \sigma(s(x_0), g_0 \exp(tA)) \\ &= A^\#(s(x_0) \cdot g_0)\end{aligned}$$

We have therefore proven

$$(\Psi)_{*(x_0, g_0)}(v_1, v_2) = (\sigma_{g_0})_{*s(x_0)}(s_{*x_0}(v_1)) + A^\#(s(x_0) \cdot g_0)$$

where $A = (L_{g_0^{-1}})_{*g_0}(v_2)$. In particular, this means that every element of $T_{s(x_0)}(\mathcal{P}^{-1}(V))$ can be written as $s_{*x_0}(v) + A^\#(s(x_0))$ for some $A \in \mathcal{G}$. For every $j \in J$ we define ω_j as follows: If $x_0 \in V_j$, $p = s_j(x_0)$, $v \in T_{x_0}(X)$ and $A \in \mathcal{G}$, we define

$$\omega_j(p)((s_j)_{*x_0}(v) + A^\#(s(x_0))) = \mathcal{A}_j(x_0)(v) + A$$

Now, since any point in $\mathcal{P}^{-1}(V_j)$ is given by $p \cdot g$ for some $g \in G$, we set for any $w \in T_{p \cdot g}(\mathcal{P}^{-1}(V_j))$

$$\omega(p \cdot g)(w) = ad_{g^{-1}} \circ \omega_j((\sigma_{g^{-1}})_{*p \cdot g}(w))$$

A trivial computation shows that ω_j satisfies the first connection property, i.e. we have

$$\omega(p \cdot g)((\sigma_{g'})_{*p \cdot g'}(v)) = ad_{(g')^{-1}} \circ \omega_j((p \cdot g) \cdot (g')^{-1})$$

For the second property, we use Lemma 3.3.14:

$$\begin{aligned} \omega_j(p \cdot g)(A^\#) &= ad_{g^{-1}} \circ \omega_j(p)((\sigma_{g^{-1}})_{*p \cdot g}(A^\#)) \\ &= ad_{g^{-1}} \circ \omega_j(p)((ad_g(A))^\#(p \cdot g)) \\ &= ad_{g^{-1}} \circ (ad_g(A)) \\ &= A \end{aligned}$$

So we have proven that ω_j is a connection form on the bundle $\mathcal{P} : \mathcal{P}^{-1}(V_j) \rightarrow V_j$.

Let (V_i, Ψ_i) and (V_j, Ψ_j) be two trivializations with $x_0 \in V_i \cap V_j$ and $v \in T_{xx_0}(X)$. We then have

$$\begin{aligned} s_j(x_0) &= s_i(x_0) \cdot g_{ij}(x_0) = \sigma \circ (s_i, g_{ij})(x_0) \\ \Rightarrow (s_j)_{*x_0}(v) &= (\sigma_{g_{ij}(x_0)})_{*s_i(x_0)}((s_i)_{*x_0}(v)) + (\sigma_2 \circ g_{ij})_{*x_0}(v) \end{aligned}$$

It follows from an easy computation that

$$(\sigma_2 \circ g_{ij})_{*x_0}(v) = \frac{d}{dt}|_0 \sigma(s_j(x_0), \exp(t\Theta_{ij}(x_0)(v))) = (\Theta_{ij}(x_0)(v))^\#(s_j(x_0))$$

So that we have proven that

$$(s_j)_{*x_0}(v) = (\sigma_{g_{ij}(x_0)})_{*s_i(x_0)}((s_i)_{*x_0}(v)) + (\Theta_{ij}(x_0)(v))^\#(s_j(x_0))$$

We now show that on $\mathcal{P}^{-1}(V_i \cap V_j)$, we have $\omega_j = \omega_i$. It is actually enough to prove that they agree on $s_j(V_j \cap V_i)$. The result is trivial if $v = A^\#(p)$. The other case goes as follows:

$$\begin{aligned} \omega_i(p)((s_j)_{*x_0}(v)) &= \omega_i(p)((\sigma_{g_{ij}(x_0)})_{*s_i(x_0)}((s_i)_{*x_0}(v)) + (\Theta_{ij}(x_0)(v))^\#(s_j(x_0))) \\ &= \omega_i(p)((\sigma_{g_{ij}(x_0)})_{*p \cdot (g_{ij}(x_0))^{-1}}((s_i)_{*x_0}(v)) + \Theta_{ij}(x_0)(v)) \\ &= ad_{(g_{ij}(x_0))^{-1}} \circ \omega_j((s_i)_{*x_0}(v)) + \Theta_{ij}(x_0)(v) \\ &= ad_{(g_{ij}(x_0))^{-1}} \circ \mathcal{A}_i(x_0)(v) + \Theta_{ij}(x_0)(v) \\ &= \mathcal{A}_j(x_0)(v) \\ &= \omega_j(p)((s_j)_{*x_0}(v)) \end{aligned}$$

Finally, we define ω on P by $\omega|_{\mathcal{P}^{-1}(V_j)} = \omega_j$ for every $j \in J$. Then ω is well-defined and has all of the required properties. $\square$

We now state a trivial corollary of this theorem, but which will have some use later on.

Corollary 4.1.1. *Any $\mathcal{G}$ -valued 1-form on the manifold X is the pullback to X of a unique connection form on the trivial $\mathcal{G}$ -bundle over X .*

Proof. Simply choose a global trivialization $\{(X, \Psi)\}$ in which case the condition of the theorem is trivially verified. $\square$

Definition 4.1.3. *Let ω be a connection form on P . For each $p \in P$, we define the horizontal subspace $Hor_p(P)$ of $T_p(P)$ determined by ω as*

$$Hor_p(P) = \{v \in T_p(P) | \omega_p(v) = 0\}$$

Lemma 4.1.4. *Let ω be a connection form on P . The tangent space at any point $p \in P$ can then be separated as*

$$T_p(P) = Hor_p(P) \oplus Vert_p(P)$$

Proof. We start by proving that the intersection of the horizontal and vertical subspaces is empty. To do so, suppose v belongs to that intersection, then by Lemma 3.3.13, there exists an $A \in \mathcal{G}$ such that $v = A^\#(p)$. Furthermore, since v is horizontal, we have

$$\omega_p(v) = 0 = \omega_p(A^\#(p)) = A$$

So that $v = 0$, which proves that the intersection is empty.

It is therefore sufficient to prove that $\dim \text{Hor}_p(P) + \dim \text{Vert}_p(P) = \dim T_p(P)$. Since ω_p is a linear transformation of $T_p(P)$ to $\mathcal{G}$, we have that

$$\dim T_p(P) = \dim \ker \omega + \dim \text{Im} \omega$$

But since the kernel of ω is just the horizontal subspace and ω_p is surjective onto $\mathcal{G}$ since for any $A \in \mathcal{G}$, we have $\omega_p(A^\#(p)) = A$, so that the dimension of the image is $\dim \mathcal{G} = \dim \text{Vert}_p(P)$, we obtain the result. $\square$

Remark 4.1.1. For convenience, we state the dimensions of the different spaces:

1. $\dim T_p(P) = \dim P$
2. $\dim \text{Vert}_p(P) = \dim \mathcal{G} = \dim G$
3. $\dim \text{Hor}_p(P) = \dim X$: This follows from $\dim P = \dim X + \dim G$ and the decomposition of the previous lemma.

It is also interesting to mention that this implies that the projection $\mathcal{P}$ is a submersion.

The following lemma gives us a deeper understanding of the horizontal subspace by relating it to the tangent space of the base space.

Lemma 4.1.5. Let $\mathcal{P}(p) = x$. Then $\mathcal{P}_*p$ is identically zero on $\text{Vert}_p(P)$ and is an isomorphism from $\text{Hor}_p(P)$ onto $T_x(X)$.

Lemma 4.1.6. The horizontal subspaces determined by the connection form ω are invariant under the action of G on P . More precisely, this means that

$$(\sigma_g)_*p(\text{Hor}_p(P)) = \text{Hor}_{p \cdot g}(P) \quad \forall g \in G, \forall p \in P$$

Proof. Let $v \in \text{Hor}_p(P)$. We then have

$$\omega_{p \cdot g}((\sigma_g)_*p(v)) = \omega_{p \cdot g}((\sigma_g)_*(p \cdot g \cdot g^{-1})(v)) = \text{ad}_{g^{-1}}(\omega_p(v))$$

So that $(\sigma_g)_*p(\text{Hor}_p(P)) \subseteq \text{Hor}_{p \cdot g}(P)$.

Now suppose $w \in \text{Hor}_{p \cdot g}(P)$. Since σ_g is a diffeomorphism, its pushforward is an isomorphism, so there exists a $v \in T_p(P)$ such that

$$w = (\sigma_g)_*p(v)$$

Furthermore, we have

$$\omega_p(v) = \omega_p((\sigma_{g^{-1}})_*p \cdot g(w)) = \text{ad}_g(\omega_{p \cdot g}(w)) = 0$$

So that $(\sigma_g)_*p(\text{Hor}_p(P)) \supseteq \text{Hor}_{p \cdot g}(P)$. $\square$

We now introduce the notion of distribution.

Definition 4.1.4. Let P be a smooth manifold of dimension $n+k$. An n -dimensional distribution $\mathcal{D}$ on P is an assignment to each $p \in P$ of an n -dimensional subspace $\mathcal{D}(p)$ of $T_p(P)$ that is smooth in the sense that, for each $q \in P$, there exists a neighborhood U of q and n smooth vector fields $\mathbf{V}_1, \dots, \mathbf{V}_n$ on U such that, for every $p \in U$, $\mathcal{D}(p)$ is spanned by $\{\mathbf{V}_1, \dots, \mathbf{V}_n\}$.

Proposition 4.1.1. Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X with a connection form ω and with $\dim X = n$ and $\dim G = k$. Then the assignment $p \rightarrow \text{Hor}_p(P)$ is an n -dimensional distribution.

Proof. We only need to prove smoothness as defined above, since the rest has been proven in the previous lemmas. We fix $q \in P$ and choose $n + k$ smooth vector fields $\mathbf{W}_1, \dots, \mathbf{W}_{n+k}$ on a neighborhood U of q such that, for each $p \in U$, they span $T_p(P)$ (e.g. coordinate vector fields). Furthermore, let $A_1, \dots, A_k$ be a basis for $\mathcal{G}$, so that we can write $\omega = \omega^j A_j$ for some smooth $\mathbb{R}$ -valued 1-forms ω^j . We define for all $i = 1, \dots, n$

$$\mathbf{V}_i = \mathbf{W}_i - \omega^j(\mathbf{W}_i)A_j^\#$$

These are clear smooth. Furthermore, we have

$$\omega(\mathbf{V}_i) = \omega(\mathbf{W}_i) - \omega^j(\mathbf{W}_i)\omega(A_j^\#) = \omega(\mathbf{W}_i) - \omega^j(\mathbf{W}_i)A_j = 0$$

So the V_i are horizontal. It simple to show that for every $p \in U$, $\{V_1(p), \dots, V_n(p)\}$ spans $\text{Hor}_p(P)$. $\square$

Using these result, we can reverse the process that led us from the connection form to the distribution.

Proposition 4.1.2. *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X with $\dim X = n$ and $\dim G = k$. Let $\mathcal{D}$ be a smooth n -dimensional distribution which satisfies*

$$T_p(P) = \mathcal{D}(p) \oplus \text{Vert}_p(P) \quad \text{and} \quad (\sigma_g)_{*p}(\mathcal{D}(p)) = \mathcal{D}(p \cdot g) \quad \forall p \in P, \forall g \in G$$

If we defined a $\mathcal{G}$ -valued connection form for every $p \in P$ by

$$\omega_p(v + A^\#(p)) = A$$

where $v \in \mathcal{D}(p)$, then ω is a connection form on the bundle and $\text{Hor}_p(P) = \mathcal{D}(p)$ for every $p \in P$.

4.2 Gauge equivalence

Theorem 4.2.1. *Let $\mathcal{B}_1 = (P_1, \mathcal{P}_1, \sigma^1)$ and $\mathcal{B}_2 = (P_2, \mathcal{P}_2, \sigma^2)$ be two smooth principal G -bundles over the same base space X , $f : P_1 \rightarrow P_2$ a smooth bundle map and ω a connection form on P_2 . Then $f^*\omega$ is a connection form on P_1 .*

Proof. We will use the notation $A^\#$ for the fundamental vector fields on both P_1 and P_2 . We want to prove

$$(\sigma_g^1)^*(f^*\omega) = \text{ad}_{g^{-1}} \circ (f^*\omega) \quad \text{and} \quad (f^*\omega)(A^\#) = A \quad \forall p \in P, \forall g \in G$$

f is a bundle map so that $f \circ \sigma_g^1 = \sigma_g^2 \circ f$, which implies

$$(\sigma_g^1)^*(f^*\omega) = (\sigma_g^2 \circ f)^*\omega$$

Now let $p \in P_1$ and $v \in T_p(P_1)$,

$$\begin{aligned} ((\sigma_g^2 \circ f)^*\omega)_p(v) &= \omega_{f(p) \cdot g}((\sigma_g^2)_{*(f(p) \cdot g)}(f_{*p}(v))) \\ &= (\text{ad}_{g^{-1}} \circ (f^*\omega))_p(v) \end{aligned}$$

which proves the first connection property. For the second, we observe by unfolding the definitions and using the bundle map property that

$$f_{*p}(A^\#(p)) = A^\#(f(p))$$

The second property is a direct consequence of this. $\square$

Lemma 4.2.1. *There is a one-to-one correspondence between local gauge transformations on $V \subseteq X$ and automorphisms of the bundle $\mathcal{P}^{-1}(V)$.*

Proof. A local gauge transformation is a change in cross-section, which arises through the group action. More precisely, let $s : V \rightarrow \mathcal{P}^{-1}(V)$ be a local cross-section and $g : V \rightarrow G$ a smooth map. We now define $s^g : V \rightarrow \mathcal{P}^{-1}(V)$ by

$$s^g(x) = s(x) \cdot g(x) \quad \forall x \in V$$

Since $\mathcal{P}(s^g(x)) = \mathcal{P}(s(x)) = \text{id}_V$, s^g also is a local cross-section so that

$$\mathcal{P}^{-1}(V) = \bigcup_{x \in V} s^g(x) \cdot G$$

We can therefore define a map $f : \mathcal{P}^{-1}(V) \rightarrow \mathcal{P}^{-1}(V)$ by

$$f(s(x) \cdot h) = s^g(x) \cdot h$$

We now show that f is a smooth automorphism of the G -bundle $\mathcal{P}^{-1}(V)$. Smoothness comes trivially from composition. Furthermore, we have

$$f((s^g(x) \cdot h) \cdot g) = f(s^g(x) \cdot (hg)) = (s^g(x) \cdot h) \cdot g = f(s^g(x) \cdot h) \cdot g$$

We also trivially have $\mathcal{P} \circ f = \mathcal{P}$, which proves that the induced map is the identity.

Conversely, suppose we are given an automorphism f of $\mathcal{P}^{-1}(V)$ onto itself. Let $s : V \rightarrow \mathcal{P}^{-1}(V)$ be any cross-section and let us define the map $s' : V \rightarrow \mathcal{P}^{-1}(V)$ by

$$s'(x) = f^{-1}(s(x)) \quad \forall x \in V$$

We have

$$\mathcal{P}(s'(x)) = (\mathcal{P} \circ f^{-1})(s(x)) = \mathcal{P}(s(x)) = x$$

so that s' is a cross-section. Thus, there exists a unique $g(x) \in G$ such that

$$s'(x) = s(x) \cdot g(x)$$

But $g(x)$ is smooth since $f^{-1} \circ s = \sigma \circ (s, g)$ in which everything is smooth.

So have shown that $s' = s^g$ which concludes the proof. $\square$

Up until now, we have only dealt with local gauge transformations. The previous shows the way towards defining global gauge transformations.

Definition 4.2.1. Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X , then a global gauge transformation is an automorphism of the bundle, i.e. a diffeomorphism $f : P \rightarrow P$ of P onto itself that preserves the fiber of $\mathcal{P}$ and commutes with the action of G on P :

$$\mathcal{P} \circ f = \mathcal{P} \quad \text{and} \quad f(p \cdot g) = f(p) \cdot g$$

Since compositions and inverses of automorphisms are also automorphisms, the collection of all gauge transformations of a bundle is a group under composition called the group of gauge transformations of $\mathcal{P} : P \rightarrow X$ and it is denoted by $\mathcal{G}(P)$.

Definition 4.2.2. We define an equivalence relation on set $\mathcal{A}(P)$ of all connections on P by imposing that two connection form ω and η on P are gauge equivalent if there exists an $f \in \mathcal{G}(P)$ such that $\eta = f^*\omega$, i.e.

$$\omega \sim \eta \quad \Leftrightarrow \quad \exists f \in \mathcal{G}(P) \text{ s.t. } \eta = f^*\omega$$

Lemma 4.2.2. Gauge equivalence is an equivalence relation on $\mathcal{A}(P)$.

Proof. Follows trivially from the composition of pushforwards and the fact that $\mathcal{G}(P)$ is a group under composition. $\square$

Definition 4.2.3. The set $\mathcal{A}(P)/\mathcal{G}(P)$ of gauge equivalence classes of connections on P is called the moduli space of connections on the bundle.

4.3 Lifting and parallel translation on principal bundles

Theorem 4.3.1. *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X and ω a connection form on P . Let α be a smooth curve in X with $\alpha(0) = x_0$ and let $p_0 \in \mathcal{P}^{-1}(x_0)$. Then there exists a unique smooth curve $\tilde{\alpha} : [0, 1] \rightarrow P$ such that*

1. $\tilde{\alpha}(0) = p_0$
2. $\mathcal{P} \circ \tilde{\alpha}(t) = \alpha$ for all $t \in [0, 1]$
3. $\tilde{\alpha}'(t) \in \text{Hor}_{\tilde{\alpha}(t)}(P)$ for all $t \in [0, 1]$

Proof. We start by assuming that $\alpha([0, 1]) \subseteq V$ for some trivialization (V, Ψ) . Let s be the associated canonical cross-section which we assume without loss of generality to satisfy $s(x_0) = p_0$. We now define

$$\tilde{\alpha}(t) = \Psi^{-1}(\alpha(t), g(t)) = s(\alpha(t)) \cdot g(t)$$

for some smooth curve g in G . For (1) to be satisfied, we need $g(0) = e$. We now proceed to show that there exists such a g so that (3) is also satisfied.

We have already shown in the proof of Theorem 4.1.1 that

$$\tilde{\alpha}'(t) = (\Psi^{-1} \circ (\alpha, g))'(t) = (\sigma_{g(t)})_{*s(\alpha(t))} (s_{*\alpha(t)}(\alpha'(t))) + A^\#(s(\alpha(t)) \cdot g(t))$$

where $A = (L_{g(t)^{-1}})_{*g(t)}(g'(t))$. We therefore have

$$\omega_{\tilde{\alpha}(t)}(\tilde{\alpha}'(t)) = \text{ad}_{g(t)^{-1}}(\beta(t)) + (L_{g(t)^{-1}})_{*g(t)}(g'(t))$$

where $\beta(t) = (s^*\omega)_{\alpha(t)}(\alpha'(t))$ is a known, smooth curve in $\mathcal{G}$. For $\tilde{\alpha}'(t)$ to be horizontal, we therefore need

$$\text{ad}_{g(t)^{-1}}(\beta(t)) = - (L_{g(t)^{-1}})_{*g(t)}(g'(t))$$

Since $L_{g(t)^{-1}}$ is a diffeomorphism, its pushforward is an isomorphism, so that we have the equivalent equation:

$$g'(t) = - (L_{g(t)})_{*e}(\text{ad}_{g(t)^{-1}}(\beta(t))) = - (R_{g(t)})_{*\beta(t)}$$

where we used the definition of the adjoint representation.

We therefore have the conclusion of the theorem if and only if $\tilde{\alpha}(t)$ has the form $\tilde{\alpha}(t) = s(\alpha(t)) \cdot g(t)$, where

$$g'(t) = - (R_{g(t)})_{*\beta(t)} \quad \text{and} \quad g(0) = e$$

where $\beta(t) = (s^*\omega)_{\alpha(t)}(\alpha'(t))$.

To prove the existence of g , we want to apply Theorem 2.1.2. Let $\delta > 0$.

We define a vector field $\mathbf{V}$ on $G \times (-\delta, 1 + \delta)$ by

$$\mathbf{V}(g, s) = \left(- (R_{g(t)})_{*\beta(t)}, \frac{d}{dt} \Big|_s \right)$$

The maximal integral curve of $\mathbf{V}$ starting at $(e, 0)$ will be of the form $(g(t), t)$ where g satisfies the sought properties, but this solution is defined on an interval $[0, t_0]$. By compactness of $\{e\} \times [0, 1]$, we can choose an $\epsilon > 0$ such that for every $r \in [0, 1]$, $\mathbf{V}_t(e, r)$ is defined for $|t| < \epsilon$. Let $(h(t), t + t_0)$ be the integral curve of $\mathbf{V}$ starting at (e, t_0) . It is defined for $|t| < \epsilon$ for some $\epsilon > 0$. If we now define $\tilde{g}(t) = h(t - t_0)g(t)$ for $|t - t_0| < \epsilon$, then $\tilde{g}(t_0) = g(t_0)$ and

$$\tilde{g}'(t) = (R_{g(t_0)})_{*h(t-t_0)}(h'(t-t_0)) = - (R_{g(t_0)})_{*h(t-t_0)}((R_{h(t-t_0)})_{*e}(\beta(t))) = - (R_{\tilde{g}(t)})_{*\beta(t)}$$

So g can be extended to a solution on $[0, t_0 + \epsilon] \cap [0, 1]$. By iterating this procedure, the solution can be extended to $[0, 1]$.

Now, let us assume that $\alpha([0, 1])$ is not contained in a single trivializing neighborhood. We have that $\alpha([0, 1])$ is compact, so that there exists a finite subcover of trivializing neighborhoods V_i . Therefore $\alpha^{-1}(V_i)$ cover the interval $[0, 1]$. Let λ be the Lesbesgue number of that cover and

then apply the same procedure as above to $I_i = [i \cdot \frac{1}{2}\lambda, (i+1) \cdot \frac{1}{2}\lambda]$ and $\alpha'_i(t) = \alpha|_{I_i}(\frac{\lambda}{2}t + i\frac{\lambda}{2})$ for i in the finite subcover. Then by construction $\alpha'_i : [0, 1] \rightarrow X$ and $\alpha'_i \subseteq V_j$ for some j . This α can then be used directly as above. Finally, we use the gluing lemma to glue all of the α'_i together and this glued α' then is the sought path. $\square$

This theorem gives us the possibility of connecting the fibers above any two points in X that can be joined by a smooth curve.

Definition 4.3.1. Let us suppose that $x_0, x_1 \in X$ and let α be a smooth curve from x_0 to x_1 . For any $p_0 \in \mathcal{P}^{-1}(x_0)$ there exists a unique smooth curve $\tilde{\alpha}_{p_0}$ in P that lifts α , goes through p_0 at $t = 0$ and whose velocity vector is horizontal at each point, so that in particular $\tilde{\alpha}_{p_0}(1) \in \mathcal{P}^{-1}(x_1)$. We then define a map $\tau_\alpha : \mathcal{P}^{-1}(x_0) \rightarrow \mathcal{P}^{-1}(x_1)$ called parallel translation along α (determined by the connection ω) by

$$\tau_\alpha(p_0) = \tilde{\alpha}_{p_0}(1)$$

The following proposition exhibits some of the properties of parallel translation.

Proposition 4.3.1. Parallel translation satisfies the following conditions:

1. τ_α commutes with the action σ of G on P , i.e.

$$\tau_\alpha \circ \sigma_g = \sigma_g \circ \tau_\alpha \quad \forall g \in G$$

2. $\tau_{\alpha^\leftarrow} = (\tau_\alpha)^{-1}$

3. Let α, β be smooth curves in X with $\alpha(1) = \beta(0)$, then

$$\tau_{\alpha\beta} = \tau_\beta \circ \tau_\alpha$$

4.4 Curvature

Definition 4.4.1. Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X and ω a connection form on P . The curvature of ω is the $\mathcal{G}$ -valued 2-form Ω on P defined by

$$\Omega(p)(v, w) = \Omega_p(v, w) = (d\omega)_p(v^H, w^H)$$

for every $p \in P$, every $v, w \in T_p(P)$ and where the superscript H denotes the horizontal part of the corresponding vector, i.e. its projection onto the horizontal subspace of $T_p(P)$.

Lemma 4.4.1. The curvature is smooth, bilinear and skew-symmetric.

Proof. Smoothness is trivial. To prove that the kernel is linear subspace, i.e. that $v_1^H + v_2^H \in \text{Hor}_p(P)$. We have

$$v_1 + v_2 = (v_1 + v_2)^H + (v_1 + v_2)^V$$

but since $v_i = v_i^H + v_i^V$ and $v_1 + v_2$ can be uniquely decomposed in terms of horizontal and vertical vectors, we have that

$$(v_1 + v_2)^H = v_1^H + v_2^H$$

Now choose a basis $\{e_i\}$ of $\mathcal{G}$. Since the components of $d\omega$ relative to this basis are smooth, it will suffice to prove that for every smooth vector field $\mathbf{V}$, the horizontally projected vector field defined by $V^H(p) = (V(p))^H$ is also smooth. We have

$$\omega(V - \omega^i(V)e_i^\#) = 0 \Rightarrow V - \omega^i(V)e_i^\# \in \text{Hor}_p(P)$$

and $e_i^\#$ is vertical so that $\omega^i(V)e_i^\#$ also is vertical, therefore since

$$V = V - \omega^i(V)e_i^\# + \omega^i(V)e_i^\#$$

By uniqueness of the decomposition, we have that

$$V^H = V - \omega^i(V)e_i^\#$$

$\square$

Theorem 4.4.1 (Cartan structure equation). *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X and ω a connection form on P . Let Ω be the curvature of ω . Then*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

Proof. Fix a $p \in P$ and $v, w \in T_p(P)$. Recall that the wedge product on $\mathcal{G}$ is determined by the Lie bracket, i.e. we have

$$[\omega, \omega]_p(v, w) = 2[\omega_p(v), \omega_p(w)]$$

So what we need to prove is

$$(d\omega)_p(v, w) = -[\omega_p(v), \omega_p(w)] + \Omega_p(v, w)$$

Using the decomposition of the tangent into horizontal and vertical subspaces and the bilinearity and skew-symmetry of the wedge product, the exterior derivative and the curvature, it will suffice to verify three case:

1. v and w are both horizontal: Since the vectors are horizontal, they are in the kernel of ω_p so that

$$[\omega_p(v), \omega_p(w)] = 0$$

So that

$$\Omega_p(v, w) = (d\omega)_p(v^H, w^H) = (d\omega)_p(v, w)$$

2. v and w are both vertical: Since the vectors are vertical, their horizontal projection is zero so that

$$\Omega_p(v, w) = 0$$

There exist $A, B \in \mathcal{G}$ so that $v = A^\#(p)$ and $w = B^\#(p)$. Since $\omega(A^\#)$ and $\omega(B^\#)$ are constant functions, we have that

$$(d\omega)_p(v, w) = (d\omega)_p(A^\#, B^\#) = -\omega_p([A^\#, B^\#])$$

By the naturality of the fundamental vector field (see Theorem 3.3.4), we have:

$$(d\omega)_p(v, w) = \omega_p([A^\#, B^\#]) = [\omega_p(A^\#), \omega_p(B^\#)] = [\omega_p(v), \omega_p(w)]$$

3. v is vertical and w is horizontal: We then have

$$\Omega_p(v, w) = (d\omega)_p(0, w) = 0 \quad \text{and} \quad -[\omega_p(v), \omega_p(w)] = -[\omega_p(v), 0] = 0$$

We write $v = A^\#(p)$ for some $A \in \mathcal{G}$.

Let $W \in \mathcal{X}(P)$ such that $W(p) = w \in \text{Hor}_p(P)$. We have already proven that

$$W^H = W - \omega^i(W)e_i^\#$$

Furthermore, we have that $W^H(p) = w$ since $e_i^\#$ is vertical. So if we define $W \equiv W^H$, we have a horizontal vector field with starting vector w .

Thus, we obtain

$$d\omega(A^\#, W) = A^\#(\omega(W)) - W(\omega(A^\#)) - \omega([A^\#, W]) = -\omega([A^\#, W])$$

since $\omega(W)$ is zero and $\omega(A^\#) = A$ is constant. We can conclude the proof by proving that $[A^\#, W]$ is horizontal. To do so, we use the formula

$$[A^\#, W] = \lim_{t \rightarrow 0} \frac{1}{t} \left(\left(A_{-t}^\# \right)_{*\alpha_p(t)} (W_{\alpha_p(t)}) - W_p \right)$$

where $\alpha_p(t) = p \cdot \exp(tA)$ and $A_{-t}^\# = \sigma_{\exp(-tA)}$ by Theorem 3.3.4. We therefore have that

$$\left(A_{-t}^\#\right)_{*\alpha_p(t)} = (\sigma_{\exp(-tA)})_{*p \cdot \exp(tA)}$$

which by Lemma 4.1.6 carries $\text{Hor}_{\alpha_p(t)}(P)$ onto $\text{Hor}_p(P)$, so that $\left(A_{-t}^\#\right)_{*\alpha_p(t)}(W_{\alpha_p(t)})$ is horizontal and by linearity of the horizontal subspace, we have

$$\frac{1}{t} \left(\left(A_{-t}^\#\right)_{*\alpha_p(t)}(W_{\alpha_p(t)}) - W_p \right)$$

is horizontal.

To conclude, we use the formula 2.1.5. The components of ω are all continuous, so that we can take the limit outside and then ω is acting on horizontal vectors, which prove that $[A^\#, W]_p$ is in the kernel.

□

Definition 4.4.2. The pullback $s^*\Omega$ of the curvature Ω is called the local field strength in gauge s and is often denoted $\mathcal{F}_s$ or simply $\mathcal{F}$.

The following lemmas gives us a local version of the Cartan structure equation.

Lemma 4.4.2. Let ω be a connection 1-form on a principal G -bundle $\mathcal{B} = (P, \mathcal{P}, \sigma)$ over X with curvature Ω . Then, for every $g \in G$, we have

$$\sigma_g^* \Omega = \text{ad}_{g^{-1}} \circ \Omega$$

The equivalent of the Cartan structure equation is the given by

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$$

where $\mathcal{F} = s^*\Omega$ and $\mathcal{A} = s^*\omega$.

Finally, the same formula holds in local coordinates

$$(s \circ \phi^{-1})^* \Omega = d \left((s \circ \phi^{-1})^* \omega \right) + \frac{1}{2} [(s \circ \phi^{-1})^* \omega, (s \circ \phi^{-1})^* \omega]$$

where (ϕ, V) is a chart for X .

Proof. From the structure equation, we have:

$$\sigma_g^* \Omega = d(\sigma_g^* \omega) + \frac{1}{2} \sigma_g^*([\omega, \omega]) = d(\text{ad}_{g^{-1}} \circ \omega) + \frac{1}{2} \sigma_g^*([\omega, \omega])$$

The rest follows from trivial computations using the fact the definition of the exterior derivative and the naturality of the pullback with respect to the the exterior derivative as well as to the Lie bracket.

By shrinking V appropriately, one can assume that V is a coordinate neighborhood, so that

$$\begin{aligned} (s \circ \phi^{-1})^* \Omega &= (\phi^{-1})^* (s^* \omega) \\ &= d \left((s \circ \phi^{-1})^* \omega \right) + \frac{1}{2} [(s \circ \phi^{-1})^* \omega, (s \circ \phi^{-1})^* \omega] \end{aligned}$$

where we used the naturality of the pullback with respect to the Lie bracket and the exterior derivative. □

We now exhibit a expression for the field strength in term of local coordinates.

Lemma 4.4.3. Let (V, ψ) be a chart with coordinate functions $x^1, \dots, x^n$. On V , we have

$$\mathcal{A} = \mathcal{A}_\alpha dx^\alpha \quad \text{and} \quad \mathcal{F} = \frac{1}{2} \mathcal{F}_{\alpha\beta} dx^\alpha \wedge dx^\beta$$

where the $\mathcal{A}_\alpha$ and $\mathcal{F}_{\alpha\beta}$ are $\mathcal{G}$ -valued functions on V . We then have

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha + [\mathcal{A}_\alpha, \mathcal{A}_\beta]$$

where ∂_α stand for $\frac{\partial}{\partial x^\alpha}$ and these derivatives are computed componentwise in $\mathcal{G}$.

Proof. Follows from a simple calculation. $\square$

One interesting property of the local field strength is the simple form of local gauge transformations.

Theorem 4.4.2. Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X and ω a connection form on P . Let Ω be the curvature of ω . Let $s_i : V_i \rightarrow \mathcal{P}^{-1}(V_i)$ and $s_j : V_j \rightarrow \mathcal{P}^{-1}(V_j)$ be two local cross-sections with $V_i \cap V_j \neq \emptyset$ and let $g_{ij} : V_i \cap V_j \rightarrow G$ be the corresponding transition function. Then on $V_i \cap V_j$, we have

$$s_j^* \Omega = ad_{g_{ij}^{-1}} \circ s_i^* \Omega \quad \Leftrightarrow \quad \mathcal{F}_j = ad_{g_{ij}^{-1}} \mathcal{F}_i$$

where $\mathcal{F}_k = s_k^* \mathcal{F}$. In particular, if G is a matrix group, we have

$$\mathcal{F}_j = g_{ij}^{-1} \mathcal{F}_i g_{ij}$$

Finally, in local coordinates, the formula becomes

$$(s_j \circ \psi^{-1})^* \Omega = ad_{g_{ij}^{-1} \circ \psi^{-1}} \circ (s_i \circ \psi^{-1})^* \Omega$$

on U where (U, ψ) is a chart with $U \subseteq V_i \cap V_j$.

Proof. We fix $x_0 \in V_i \cap V_j$ and $v, w \in T_{x_0}(X)$. We then have

$$(s_j^* \Omega)_{x_0}(v, w) = \Omega_{s_j(x_0)}((s_j)_* x_0(v), (s_j)_* x_0(w))$$

In Theorem 4.1.1, we have proven

$$(s_j)_* x_0(v) = (\sigma_{g_{ij}(x_0)})_* s_i(x_0)((s_i)_* x_0(v)) + [\Theta_{ij}(v)]^\#(s_j(x_0))$$

Now, since only the horizontal parts plays a role for the curvature, we have

$$\begin{aligned} (s_j^* \Omega)_{x_0}(v, w) &= \Omega_{s_j(x_0)}((\sigma_{g_{ij}(x_0)})_* s_i(x_0)((s_i)_* x_0(v)), (\sigma_{g_{ij}(x_0)})_* s_i(x_0)((s_i)_* x_0(w))) \\ &= \left(\sigma_{g_{ij}(x_0)}^* \Omega \right)_{s_i(x_0)}((s_i)_* x_0(v), (s_i)_* x_0(w)) \\ &= ad_{g_{ij}(x_0)^{-1}} \circ \Omega_{s_i(x_0)}((s_i)_* x_0(v), (s_i)_* x_0(w)) \\ &= ad_{g_{ij}(x_0)^{-1}} \circ (s_i^* \Omega)_{x_0}(v, w) \end{aligned}$$

So we have proven the first formula.

The second formula follows from a simple computation using the definition of the adjoint map. $\square$

Proposition 4.4.1. Let $\mathcal{B}_1 = (P_1, \mathcal{P}_1, \sigma^1)$ and $\mathcal{B}_2 = (P_2, \mathcal{P}_2, \sigma^2)$ be two smooth principal G -bundles over the same base space X , $f : P_1 \rightarrow P_2$ a smooth bundle map and ω a connection form on P_2 with curvature Ω . Then the curvature of $f^* \omega$ is $f^* \Omega$.

Proof. The only thing one needs to prove is that $(f_{*p_1}(v))^H = f_{*p_1}(v^H)$. By linearity of the pushforward, we have that

$$f_{*p_1}(v) = f_{*p_1}(v^H) + f_{*p_1}(v^V)$$

and

$$\omega(f_{*p_1}(v^H)) = (f^* \omega)_{p_1}(v^H) = 0 \Rightarrow f_{*p_1}(v^H) \text{ is horizontal}$$

Finally, $f_{*p_1}(v^V) = f_{*p_1}(A^\#(p_1))$ for some $A \in \mathcal{G}$ and it is simple to verify that $f_{*p_1}(A^\#(p_1)) = A^\#(f(p_2))$ so that $f_{*p_1}(v^V)$ is vertical, which proves the claim. $\square$

Corollary 4.4.1. *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a smooth principal G -bundle over X . Let ω and η be two gauge equivalent connection 1-forms. Then the corresponding field strengths are also related by the adjoint representation as in Theorem 4.4.2.*

Proof. Since $\omega \sim \eta$, there exists a bundle automorphism f such that $\eta = f^*\omega$. Now, let $s : V \rightarrow \mathcal{P}^{-1}(V)$ with respect to which we compute the field strength relative to ω , i.e. $\mathcal{F}^\omega = s^*\Omega$. So we then get

$$\mathcal{F}^\eta = s^*(f^*\Omega) = (f \circ s)^*\Omega$$

But $f \circ s$ is also a cross-section, so both field strengths are a pullback of the same curvature Ω . Theorem 4.4.2 then tells us they are related by the adjoint representation where the transition function appearing in the transformation simply is the transition function relating the two cross-sections s and $f \circ s$. $\square$

Definition 4.4.3. *A gauge potential of the form*

$$\mathcal{A} = g^*\Theta$$

where Θ is the Cartan 1-form and $g : V \rightarrow G$ is a smooth function defined on an open subset of X is said to be pure gauge.

Lemma 4.4.4. *A pure gauge potential has zero field strength.*

Proof. By the transformation law for gauge potentials, we know that a pure gauge potential is the gauge transform by g of zero. Furthermore, the form gauge transformations for field strengths implies that if there exists a local gauge in which the field strength is zero, then the field strength is zero in any other gauge. The result then follows. $\square$

Definition 4.4.4. *A connection ω on a principal bundle $\mathcal{P} : P \rightarrow X$ is said to be flat if its curvature 2-form Ω is identically zero.*

Example 4.4.1. *Let $P = X \times G$ be the trivial G -bundle over X , π the projection onto the second factor and Θ the Cartan 1-form on G , then*

$$\omega = \pi^*\Theta$$

is a flat connection form, which can easily be verified.

Let us compute the curvature:

$$\begin{aligned} d\omega &= \pi^*(d\Theta) \\ &= \pi^*\left(-\frac{1}{2}[\Theta, \Theta]\right) \\ &= -\frac{1}{2}[\omega, \omega] \\ \Rightarrow \Omega &= d\omega + \frac{1}{2}[\omega, \omega] = 0 \end{aligned}$$

where we used the structure equation for G .

Part II

The Applications

5 A word on the homotopy groups of spheres

In this section, we give a complete proof for the fundamental group of circle S^1 . We then give an outline of the proof the higher homotopy groups of the higher-dimensional spheres.

5.1 Fundamental group of S^1

In this section, we use the covering space $\mathcal{P} : \mathbb{R} \rightarrow S^1$ where $\mathcal{P}(s) = e^{2\pi si}$.

Definition 5.1.1. Let α be a loop in S^1 at $1 = e^{2\pi 0i}$. We then define the degree of α by

$$\text{Deg}(\alpha) = \tilde{\alpha}(1)$$

Lemma 5.1.1. Let α be a loop in S^1 at $1 = e^{2\pi 0i}$. Then $\text{Deg}(\alpha) \in \mathbb{Z}$.

Proof. By the unique lifting theorem, α lifts uniquely to $\tilde{\alpha}$ in $\mathbb{R}$ such that $\tilde{\alpha}(0) = 0$. Furthermore, we also have that

$$\mathcal{P} \circ \tilde{\alpha}(1) = \alpha(1) = 1 \Rightarrow \tilde{\alpha}(1) \in \mathcal{P}^{-1}(1)$$

Since the fibers of our covering space are all homeomorphic to $\mathbb{Z}$, we have proven the result. $\square$

So the degree of a loop is essentially the number of times it wraps itself around S^1 . The degree has one more crucial property.

Lemma 5.1.2. The degree of a loop is a homotopy invariant, i.e. if $\alpha \simeq \alpha'$ rel $\{0, 1\}$ where α' is another loop in S^1 at 1, then $\text{Deg}(\alpha) = \text{Deg}(\alpha')$.

Proof. This is a direct consequence of Corollary 3.2.2. $\square$

So that the degree actually defines a map from the fundamental group to the integers. It turns out that this map is an isomorphism.

Theorem 5.1.1. Let $\text{deg} : \pi_1(S^1) \rightarrow \mathbb{Z}$ be defined by $\text{deg}([\alpha]) = \tilde{\alpha}(1)$ where $\tilde{\alpha}$ is the unique lift of α such that $\tilde{\alpha}(0) = 0$. This map is an isomorphism of groups, so that in particular we have $\pi_1(S^1) \simeq \mathbb{Z}$.

Proof. Let $[\alpha], [\beta] \in \pi_1(S^1, 1)$ and $\text{deg}(\alpha) = m$ and $\text{deg}(\beta) = n$, so that $\text{deg}([\alpha]) + \text{deg}([\beta]) = m + n$. We want to show that

$$\text{deg}([\alpha][\beta]) = m + n$$

To do so, we must compute the lift of $\alpha\beta$ to $\mathbb{R}$ which starts at $0 \in \mathbb{R}$. We define γ a path in $\mathbb{R}$ by

$$\gamma(s) = m + \tilde{\beta}(s)$$

which is then a path from m to $m + n$ with the property that

$$\begin{aligned} \mathcal{P} \circ \gamma(s) &= \exp[2\pi(m + \tilde{\beta}(s))i] \\ &= \mathcal{P}(\tilde{\beta}(s)) \\ &= \beta(s) \end{aligned}$$

We therefore obtain

$$\begin{aligned} \mathcal{P} \circ (\tilde{\alpha}\gamma)(s) &= \begin{cases} \mathcal{P}(\tilde{\alpha}(2s)) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \mathcal{P}(\gamma(2s - 1)) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \beta(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= (\alpha\beta)(s) \end{aligned}$$

We have therefore proven that $\tilde{\gamma}$ is a lift to $\mathbb{R}$ of $\alpha\beta$ which starts at $(\tilde{\alpha}\gamma)(0) = \tilde{\alpha}(0) = 0 \in \mathbb{R}$. We therefore conclude that by uniqueness that $\tilde{\alpha}\gamma = \tilde{\alpha}\beta$. We have obtained

$$\deg([\alpha][\beta]) = \tilde{\alpha}\gamma(1) = m + \tilde{\beta}(1) = m + n$$

which proves that $\deg$ is a homomorphism.

We now prove that $\deg$ is a bijection. Let $n \in \mathbb{Z}$ and define $\tilde{\alpha} : [0, 1] \rightarrow \mathbb{R}$ by

$$\tilde{\alpha}(s) = ns$$

We then have

$$\mathcal{P} \circ \tilde{\alpha}(s) = \exp(2\pi nsi)$$

which is a loop at $1 \in S^1$ with degree $\deg([\mathcal{P} \circ \tilde{\alpha}]) = n$. So we have proven that $\deg$ is onto.

To prove injectivity, we prove that $\deg$ has trivial kernel. Suppose $\deg([\alpha]) = 0$ so that $\tilde{\alpha}$ is a loop at 0 in $\mathbb{R}$. Since $\mathbb{R}$ is simply connected, we have

$$\tilde{\alpha} \simeq 0 \text{ rel } \{0, 1\} \Rightarrow [\tilde{\alpha}] = [0] \in \pi_1(\mathbb{R}, 0)$$

Now, we apply the induced homomorphism $\mathcal{P}_* : \pi_1(\mathbb{R}, 0) \rightarrow \pi_1(S^1, 1)$ which gives us

$$\mathcal{P}_*([\tilde{\alpha}]) = \mathcal{P}([0]) \Rightarrow [\alpha] \equiv [\mathcal{P} \circ \tilde{\alpha}] = [\mathcal{P} \circ 0] = [1]$$

which proves that $\deg$ has trivial kernel. $\square$

Corollary 5.1.1. *For any $n \geq 2$, $\pi_n(S^1) = 0$.*

Proof. We already know that S^1 is the base space of the covering space $\mathcal{P} : \mathbb{R} \rightarrow S^1$ given by

$$\mathcal{P}(s) = \exp(2\pi si)$$

. By Theorem 3.2.2, we have $\pi_n(\mathbb{R}) \simeq \pi_n(S^1)$. But $\mathbb{R}$ is simply connected so that $\pi_n(\mathbb{R}) = 0$. $\square$

5.2 Higher homotopy groups of spheres

The problem of calculating higher homotopy groups of spheres is one of the most difficult problems in topology. In this section, we only state the result for the homotopy groups of order n where n is smaller or equal to the dimension of the sphere and give a very rough outline of a possible way of calculating $\pi_n(S^n)$.

The promised is the following

$$\pi_n(S^m) \simeq \begin{cases} 0 & \text{if } n < m \\ \mathbb{Z} & \text{if } n = m \end{cases}$$

To prove $\pi_n(S^n) \simeq \mathbb{Z}$, one provides $[(S^n, N), (S^n, N)]$ with a group structure provided by the one-to-one correspondence with $\pi_n(S_n)$ established at the end of the section on higher homotopy groups. Theorem 2.3.11 then implies that two from $(S^n, N) \rightarrow (S^n, N)$ are homotopic if they are freely homotopic. As for S^1 one then associates to every map on S^n an integer called the degree of f and denoted $\deg f$. One then proves that

1. $f \simeq g \Rightarrow \deg f = \deg g$
2. $\deg f = \deg g \Rightarrow f \simeq g$: This result is the hard part and it is called Hopf's theorem.
3. $\deg(id_{S^n}) = 1$

Once these facts have been established, the result follows. More precisely, (1) proves that $\deg$ is a homomorphism, (2) proves that $\deg$ is one-to-one and (3) proves that $\deg$ since it is a homomorphism into $\mathbb{Z}$ that takes the value 1.

Of course, the definition of degree is far from as straightforward as for S^1 . In general, one uses the fact that the homology group of the sphere is isomorphic to $\mathbb{Z}$ (which is a consequence of the Mayer-Vietoris sequence), and then defines the degree of a map to be the integer multiplying the generator of the induced map, i.e. $f_* : H_n(S^n) \rightarrow H_n(S^n)$ and $[\omega]$ the generator of $H_n(S^n)$, then

$$f_*([\omega]) = m[\omega]$$

and we define $\deg f = m$. Properties (1) and (3) are then more or less straightforward, but (2) still requires quite some work.

6 Classification of principal bundles over spheres

In this section, we are going to classify the principal G -bundles over S^n where $n \geq 2$, for pathwise connected groups G , in terms of the homotopy group $\pi_{n-1}(G)$. The idea goes as follows. S^n consists of two copies of the n -dimensional disc D^n glued together at the equator which itself is a copy of S^{n-1} . The following lemma then tells us that any G -bundle over S^n to either of these discs is trivial.

Lemma 6.0.1. *Any principal G -bundle over the disc D^n is trivial.*

Proof. We have already constructed a global cross-section $s : I^N \rightarrow P$. Furthermore, we have a homeomorphism $\phi : D^n \rightarrow I^N$. Since D^n is convex, it is contractible so that there exists a homotopy F with $F(x, 0) = \phi(x)$ and $F(x, 1) = x$. Now if we define $\mathcal{P}' = \phi \circ \mathcal{P}$ and $\Psi' = \Psi \circ \phi^{-1}$, then $(P, I^n, \mathcal{P}', G)$ is a locally trivial bundle. Then as in the corollary, there exists a homotopy $\tilde{F}$ such that $\mathcal{P}' \circ \tilde{F}(x, 1) = \text{id}_{I^n}$ which implies $\mathcal{P} \circ \tilde{F}(x, 1) = \phi^{-1}(x)$. We now define $s_{D^n}(y) = \tilde{F}(\phi(y), 1)$ then

$$\mathcal{P} \circ s_{D^n}(y) = \phi^{-1}(\phi(y)) = y$$

So s_{D^n} is a global cross-section. □

We have then obtained a trivializing cover of S^n consisting of only two trivializations related by one transition function, which then determines the bundle up to equivalence. The restriction of this transition function to the equator is then a map from S^{n-1} to G . We then show that homotopic maps determine equivalent bundles, which then establishes a one-to-one correspondence between equivalence classes of G -bundles over S^n and elements of $\pi_{n-1}(G)$.

Through out this section, G will denote a pathwise connected topological group. Furthermore, we will identify S^{n-1} with the set of $x \in S^n$ such that $x^{n+1} = 0$, i.e. with the equator of S^n . We also fix some $x_0 \in S^{n-1}$ and an ϵ which satisfies $\epsilon \in (0, 1)$. We also define the following subsets:

$$\begin{aligned} D_1 &= \{x \in S^n | x^{n+1} \geq 0\} \\ D_2 &= \{x \in S^n | x^{n+1} \leq 0\} \\ V_1 &= \{x \in S^n | -\epsilon < x^{n+1} \leq 1\} \supseteq D_1 \\ V_2 &= \{x \in S^n | -1 \leq x^{n+1} < \epsilon\} \supseteq D_2 \end{aligned}$$

Thus, $V_1 \cap V_2$ is an open band in S^n containing the equator.

Lemma 6.0.2. *Let $\mathcal{B} = (P, \mathcal{P}, \sigma)$ be a principal G -bundle over S^n . Then V_1 and V_2 are trivializing neighborhoods and one can choose trivializations $\Psi_i : \mathcal{P}^{-1}(V_i) \rightarrow V_i \times G$, for $i = 1, 2$ such the transition functions g_{ij} satisfy*

$$g_{ij}(x_0) = e$$

Proof. Using Lemma 3.3.3, we start by considering the bundle $\mathcal{B}|_{\bar{V}_i}$ where $\bar{V}_i$ is the closure of V_i . Using stereographic projection, one easily proves that the $\bar{V}_i$ are homeomorphic to the disc D^n . Now, a homeomorphism is a bundle equivalence, so that $\mathcal{B}|_{\bar{V}_i}$ is equivalent to a bundle which is trivial by Lemma 6.0.1, which proves that $\mathcal{B}|_{\bar{V}_i}$ is trivial. We therefore conclude from Theorem 3.3.1 that $\mathcal{B}|_{\bar{V}_i}$ has a global cross-section. By simply restricting that cross-section to V_i , we have proven that $\mathcal{B}|_{V_i}$ have a global cross-section and are therefore trivial. We now choose two equivalences $\tilde{f}_i : \mathcal{P}^{-1}(V_i) \rightarrow V_i \times G$ where $V_i \times G$ is regarded as the bundle space of the trivial bundle over V_i .

It is trivial to verify that $(V_i, \tilde{f}_i)$ is indeed a local trivialization of $\mathcal{B}$.

Now let $\tilde{g}_{12} : V_1 \cap V_2 \rightarrow G$ be the transition function corresponding to these trivializations. We suppose that $\tilde{g}_{12}(x_0) = a \in G$. We then have by definition that

$$\tilde{\psi}_1(p)(\tilde{\psi}_2(p))^{-1} = a \quad \forall p \in \mathcal{P}^{-1}(x_0) \quad \text{and} \quad \tilde{f}_i(p) = (\mathcal{P}(p), \tilde{\psi}_i(p))$$

We now define $\psi_2 : \mathcal{P}^{-1}(V_2) \rightarrow G$ by

$$\psi_2(p) = a\tilde{\psi}_2(p) \quad \forall p \in \mathcal{P}^{-1}(V_2)$$

We then obtain

$$\tilde{\psi}_1(p)(\psi_2(p))^{-1} = aa^{-1} = e$$

We now define $\psi_i : \mathcal{P}^{-1}(V_i) \rightarrow V_i \times G$ by

$$\Psi_1 = \tilde{f}_1 \quad \text{and} \quad \Psi_2(p) = (\mathcal{P}(p), \psi_2(p))$$

Then it is trivial to verify that (V_i, Ψ_i) for $i = 1, 2$ are local trivializations for $\mathcal{B}$ with transition functions which send x_0 to e . $\square$

Henceforth, we assume that all our principal G -bundles over S^n are trivialized over V_1 and V_2 in such a way that the transition functions all carry x_0 to e .

Definition 6.0.1. *For every principal G -bundle over S^n , we therefore define the characteristic map*

$$T : (S^{n-1}, x_0) \rightarrow (G, e)$$

by

$$T = g_{12}|_{S^{n-1}}$$

Lemma 6.0.3. *Any continuous map $T : (S^{n-1}, x_0) \rightarrow (G, e)$ is the characteristic map of some principal G -bundle over S^n .*

Proof. We define $g_{11}(x) = e$ and $g_{22}(x) = e$ for all $x \in V_1$ and $x \in V_2$ respectively. Let $r : V_2 \cap V_1 \rightarrow S^{n-1}$ be a retraction of $V_2 \cap V_1$ onto the equator defined by

$$r = \sqrt{(x^i x_i)}(x^1, \dots, x^n)$$

We then define

$$g_{12}(x) = T(r(x)) \quad \forall x \in V_1 \cap V_2$$

We have obtained $g_{12}|_{S^{n-1}} = T$ and to complete the collection of transition functions, we set $g_{21} = g_{12}^{-1}$. The Reconstruction Theorem 3.3.3 then implies that there exists a principal bundle with this collection of transition, which proves our assertion. $\square$

Theorem 6.0.1. *Let G be a pathwise connected topological group and $\mathcal{B}_1$ and $\mathcal{B}_2$ two principal G -bundles over S^n where $n \geq 2$. Let T_1 and T_2 be the characteristic maps of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent if and only if $T_1 \simeq T_2 \text{ rel } \{x_0\}$.*

Proof. $\Rightarrow$: We start by assuming that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent. We also assume that they are both trivialized as described above, which also means that they have the same trivializing neighborhoods. By Lemma 3.3.12, we know that there exists continuous maps $\lambda : V_j \rightarrow G$ for $j = 1, 2$ such that

$$g_{12}^2(x) = (\lambda_1(x))^{-1} g_{12}^1(x) \lambda_2(x) \quad \forall x \in V_1 \cap V_2$$

We define

$$\mu_j = \lambda_j|_{S^{n-1}} \quad j = 1, 2$$

Thus we obtain

$$T_2(x) = (\mu_1(x))^{-1} T_1(x) \mu_2(x) \quad \forall x \in S^{n-1}$$

We have defined the T_k such that $T_1(x_0) = T_2(x_0) = e$ which implies that

$$\mu_1(x_0) = \mu_2(x_0) \equiv a \in G$$

As we have stated previously, the D_i for $i = 1, 2$ are homeomorphic to D^n , which is contractible, which in turn implies that

$$\pi_{n-1}(D_i, x_0) = 0 \quad i = 1, 2$$

Then by Lemma 2.3.10, we know that the set $[(S^{n-1}, x_0), (D_i, x_0)]$ contains only one element. This implies in particular that the inclusion map $i_i : (S^{n-1}, x_0) \hookrightarrow (D_i, x_0)$ are both homotopic to the

constant map of S^{n-1} onto x_0 , i.e. there exists for $i = 1, 2$, a homotopy $H_i : S^{n-1} \times [0, 1] \rightarrow D_i$ such that

$$H_i(x, 0) = x, H_i(x, 1) = x_0 \quad \forall x \in S^{n-1} \quad \text{and} \quad H_i(x_0, t) = x_0 \quad \forall t \in [0, 1]$$

We define $K_i = \lambda_i \circ H_i : S^{n-1} \times [0, 1] \rightarrow G$, which is then a homotopy relative to x_0 from μ_i to the constant map from S^{n-1} to G whose value is a . Finally, we define the map $K : S^{n-1} \times [0, 1] \rightarrow G$ by

$$K(x, t) = (K_1(x, t))^{-1} T_1(x) K_2(x, t)$$

We then have

$$\begin{aligned} K(x, 0) &= T_2(x) \\ K(x, 1) &= a^{-1} T_1(x) a \\ K(x_0, t) &= e \end{aligned}$$

Which proves that K is a homotopy relative to x_0 of T_2 to $a^{-1} T_1 a$.

Now it is easy to verify the following fact: let α be a path in G from a to e , then the homotopy H defined by $H(x, t) = (\alpha(t))^{-1} T_1(x) \alpha(t)$ is a homotopy relative to x_0 from $a^{-1} T_1 a$ to T_1 .

Therefore, $T_2 \simeq T_1 \text{ rel } \{x_0\}$ implies $T_2 \simeq T_1 \text{ rel } \{x_0\}$. We have therefore proven if $\mathcal{B}_1$ and $\mathcal{B}_2$, then $T_1 \simeq T_2 \text{ rel } \{x_0\}$.

$\Leftarrow$: We suppose that $T_1 \simeq T_2 \text{ rel } \{x_0\}$ and we want to show that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent. We start by proving a general result.

Lemma 6.0.4. *Let X be a topological space and G a topological group. Suppose $f, g, f', g' : (X, x_0) \rightarrow (G, e)$ with $f \simeq f' \text{ rel } \{x_0\}$ and $g \simeq g' \text{ rel } \{x_0\}$. Then*

$$fg \simeq f'g' \text{ rel } \{x_0\}$$

Proof. Follows from a simple computation. □

Using our hypothesis and the fact that $T_2^{-1} \simeq T_2^{-1} \text{ rel } x_0$, we obtain from the previous lemma that $T_1 T_2^{-1} \simeq e \text{ rel } \{x_0\}$, i.e. $T_1 T_2^{-1} : S^{n-1} \rightarrow G$ is nullhomotopic. By Lemma 2.3.7, we can continuously extend it to the disk $D^n = \{x \in S^n | x^{n+1} = 0\}$ and then compose it with the projection $P : D_1 \rightarrow D^n$ defined by

$$P(x) = (x^1, \dots, x^n, 0)$$

i.e. we have constructed a continuous extension $\nu : D_1 \rightarrow G$ of $T_1 T_2^{-1}$. We now define $\lambda_1 : V_1 \rightarrow G$ by

$$\lambda_1(x) = \begin{cases} \nu(x) & \text{if } x \in D_1 \\ g_{12}^1(x) (g_{12}^2(x))^{-1} & \text{if } x \in D_2 \cap V_1 \end{cases}$$

This map λ_1 is continuous since $D_1 \cap (D_2 \cap V_1) = S^{n-1}$ and on S^{n-1} , we have by construction

$$\nu(x) = T_1(x) (T_2(x))^{-1} = g_{12}^1(x) (g_{12}^2(x))^{-1}$$

so that continuity follows from the gluing lemma.

We now want to use Lemma 3.3.12 again to conclude that the bundles are equivalent. We have constructed things such that we can apply the Lemma in a straightforward way if $x \in D_2 \cap V_1$ but not if $x \in D_1$. We therefore can not choose simply V_2 or D_2 as second trivializing neighborhood. Instead, let $\tilde{V}_2$ be the interior of D_2 and $\tilde{\Psi}_2^i = \Psi_2^i|_{\mathcal{P}^{-1}(\tilde{V}_2)}$ for $i = 1, 2$. We now have $\{(V_1, \Psi_1^1), (\tilde{V}_2, \tilde{\Psi}_2^1)\}$ and $\{(V_1, \Psi_1^2), (\tilde{V}_2, \tilde{\Psi}_2^2)\}$ which trivialize $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Notice that the transition functions are defined on $V_1 \cap \tilde{V}_2$ which is a subset of $D_2 \cap V_1$. Finally, we define $\lambda_2 : \tilde{V}_2 \rightarrow G$ by

$$\lambda_2(x) = e \quad \forall x \in V_2$$

We have therefore obtained, for $x \in V_1 \cap \tilde{V}_2$

$$\begin{aligned} (\lambda_1(x))^{-1} g_{12}^1 \lambda_2(x) &= (\lambda_1(x))^{-1} g_{12}^1(x) \\ &= g_{12}^2(x) (g_{12}^1(x))^{-1} g_{12}^1(x) \\ &= g_{12}^2(x) \end{aligned}$$

For g_{21}^2 , we simply invert the previous equation and make use of $g_{21}^i = (g_{12}^i)^{-1}$ and if $i = j$ the condition of Lemma 3.3.12 is trivially verified. Thus we have proven that there exists continuous maps $\lambda_j : V_j \rightarrow G$ for $j = 1, 2$ such that

$$g_{ij}^2(x) = (\lambda_j(x))^{-1} g_{ij}^1(x) \lambda_i(x) \quad \forall x \in V_i \cap V_j$$

So $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent by Lemma 3.3.12. $\square$

We are now in a position to state and prove the major result of this section.

Theorem 6.0.2. *Let G be a pathwise connected topological group. Then the set of equivalence classes of principal G -bundles over S^n where $n \geq 2$ is in one-to-one correspondence with the elements of $\pi_{n-1}(G)$.*

Proof. Lemma 2.3.10 gives a one-to-one correspondence between $\pi_{n-1}(G)$ and $[(S^{n-1}, x_0), (G, e)]$. On the other hand, let us denote by $\mathcal{EB}(S^n, G)$ the set of equivalence classes of principal G -bundles over S^n and let us define the map $\gamma : \mathcal{EB}(S^n, G) \rightarrow [(S^{n-1}, x_0), (G, e)]$ by

$$\gamma([\mathcal{B}]) = T_{\mathcal{B}}$$

where $T_{\mathcal{B}}$ is the characteristic map associated to the bundle $\mathcal{B}$. First of all, this map is well-defined on $\mathcal{EB}(S^n, G)$ since if $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent, then $T_1 \equiv T_{\mathcal{B}_1}$ and $T_2 \equiv T_{\mathcal{B}_2}$ are homotopic. Additionally, Lemma 6.0.3 implies the surjectivity of γ . Finally, we have

$$\gamma([\mathcal{B}_1]) = \gamma([\mathcal{B}_2]) \Rightarrow T_1 \simeq T_2 \text{ rel } \{x_0\} \Rightarrow \mathcal{B}_1 \simeq \mathcal{B}_2$$

where the implication comes from Theorem 6.0.1. So γ is a one-to-one correspondence between $\mathcal{EB}(S^n, G)$ and $[(S^{n-1}, x_0), (G, e)]$. $\square$

7 The Hopf bundles

In this section, we will construct in quite some detail the Hopf bundles.

7.1 Projective spaces

We starting by constructing the projective spaces.

Definition 7.1.1. *Let $M^n = \mathbb{F}^n - \{0\}$ the topological subspace of $\mathbb{F}^n$. We define an equivalence relation on M^n as follows:*

$$\zeta \sim \xi \Leftrightarrow \exists a \in \mathbb{F} - \{0\} \text{ s.t. } \zeta = a\xi$$

We then define the projective space as:

$$\mathbb{FP}^{n-1} = M^n / \sim = \{[\xi] : \xi \in M^n\} = \{(\xi^1 a, \dots, \xi^n a) : \xi \in M^n, a \in \mathbb{F} - \{0\}\}$$

It will often be more convenient to view the projective spaces as quotients of spheres.

Lemma 7.1.1. *If one defines $\mathcal{P} = \mathcal{Q}|_S$, one can view the projective spaces*

$$\mathbb{FP}^{n-1} = S / \sim$$

where we remind the reader that $S = \{\xi \in \mathbb{F}^n : \langle \xi, \xi \rangle = 1\}$. Furthermore, $\mathbb{FP}^{n-1}$ also has the quotient topology determined by $\mathcal{P}$.

Proof. $\mathcal{P}$ certainly is continuous. We define $\eta : M^n \rightarrow S$ the normalization map, i.e.

$$\eta(\xi) = \frac{\xi}{\sqrt{\langle \xi, \xi \rangle}}$$

which is also continuous. We then have

$$\mathcal{Q} \circ i = \mathcal{P} \quad \text{and} \quad \mathcal{P} \circ \eta = \mathcal{Q}$$

where the second equality is simply a consequence of the equivalence relation. The second equality shows in particular that $\mathcal{P}$ is surjective. The rest is a direct consequence of the two formulas above. $\square$

Properties 7.1.1 (Fibers of $\mathbb{F}\mathbb{P}^{n-1}$). *For every $\xi \in S$, we have*

$$\mathcal{P}^{-1}([\xi]) = \{\xi a \mid a \in \mathbb{F}, |a| = 1\}$$

In particular, we have

$$1. \forall x \in S^{n-1}, \text{ we have } \mathcal{P}^{-1}([x]) = \{x, -x\} \simeq \mathbb{Z}_2$$

$$2. \forall z \in S \subseteq \mathbb{C}^n, \text{ we have } \mathcal{P}^{-1}([z]) \simeq S^1$$

$$3. \forall \xi \in S \subseteq \mathbb{H}^n, \text{ we have } \mathcal{P}^{-1}([\xi]) \simeq S^3$$

Proof. The fiber $\mathcal{P}^{-1}([\xi])$ above $[\xi] \in \mathbb{F}\mathbb{P}^{n-1}$ is simply the intersection of S and $\{\xi a : a \in \mathbb{F} - \{0\}\}$. If $\xi \in S$, then

$$\xi a \in S \Leftrightarrow \langle \xi a, \xi a \rangle = |a|^2 = 1$$

So that

$$\mathcal{P}^{-1}([\xi]) = \{\xi a \mid a \in \mathbb{F}, |a| = 1\}$$

If $\mathbb{F} = \mathbb{R}$, then $S \simeq S^{n-1}$ and

$$|a| = 1 \Leftrightarrow a = \pm 1$$

So that $\forall x \in S^{n-1}$, we have $\mathcal{P}^{-1}([x]) = \{x, -x\}$.

If $\mathbb{F} = \mathbb{C}$, then

$$|a| = 1 \Leftrightarrow a \in S^1$$

So that for any fixed $z_0 \in S \subseteq \mathbb{C}^n$, we have

$$\mathcal{P}^{-1}([z_0]) = \{(z_0^1 a, \dots, z_0^n a) \mid a \in S^1\}$$

Proving that the map $(z_0^1 a, \dots, z_0^n a) \rightarrow a$ is a homeomorphism, which implies that $\mathcal{P}^{-1}([z_0])$ is homeomorphic to S^1 is trivial.

If $\mathbb{F} = \mathbb{H}$, then

$$|a| = 1 \Leftrightarrow a \in Sp(1) \simeq S^3$$

So that for any fixed $\xi_0 \in S \subseteq \mathbb{H}^n$, we have

$$\mathcal{P}^{-1}([\xi_0]) = \{(\xi_0^1 a, \dots, \xi_0^n a) \mid a \in S^3\}$$

Proving that the map $(\xi_0^1 a, \dots, \xi_0^n a) \rightarrow a$ is a homeomorphism, which implies that $\mathcal{P}^{-1}([\xi_0])$ is homeomorphic to S^3 is trivial. $\square$

We now proceed to show the properties that make the projective spaces into topological manifolds.

Properties 7.1.2. *The projective space $\mathbb{F}\mathbb{P}^{n-1}$ are Hausdorff spaces.*

Proof. We fix $\xi_0 \in S$ and define the map $\rho : \mathbb{F}\mathbb{P}^{n-1} \rightarrow \mathbb{R}$ by

$$\rho([\xi]) = 1 - |\langle \xi, \xi_0 \rangle|^2$$

Then ρ is well-defined on $\mathbb{F}\mathbb{P}^{n-1}$ and continuous. Furthermore, it satisfies $\rho([\xi_0]) = 0$ and it is easy to see that if $[\xi] \neq [\xi_0]$, then $\rho([\xi]) \neq 0 = \rho([\xi_0])$.

Now let $[\xi_0]$ and $[\xi]$ be two distinct points in $\mathbb{F}\mathbb{P}^{n-1}$. Then $\rho([\xi_0])$ and $\rho([\xi])$ are distinct real numbers. so that there exists disjoint open intervals $I_{\xi_0} \ni \rho([\xi_0])$ and $I_\xi \ni \rho([\xi])$ in $\mathbb{R}$. By continuity of ρ , $U_{\xi_0} = \rho^{-1}(I_{\xi_0})$ and $U_\xi = \rho^{-1}(I_\xi)$ are open and disjoint sets in $\mathbb{F}\mathbb{P}^{n-1}$ which contain $[\xi_0]$ and $[\xi]$ respectively, which proves the Hausdorff property. $\square$

Properties 7.1.3. *The projective space $\mathbb{F}\mathbb{P}^{n-1}$ are locally Euclidean (i.e. homeomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}_+$).*

Proof. Since $|a| = 1$, we have that if $\xi \in S$, then $\xi \neq 0 \Leftrightarrow \xi a \neq 0$, so that for every $k = 1, \dots, n$, we can define the set

$$U_k = \{[\xi] \in \mathbb{F}\mathbb{P}^{n-1} | \xi^k \neq 0\}$$

which are open since $\mathcal{P}^{-1}(U_k) = \{\xi \in S | \xi^k \neq 0\}$ which is open in S . We define the map $\phi_k : U_k \rightarrow \mathbb{F}^{n-1}$ by

$$\phi_k([\xi]) = \left(\xi^1 (\xi^k)^{-1}, \dots, \hat{1}, \dots, \xi^n (\xi^k)^{-1} \right)$$

where the hat indicates that we delete the 1 in the k^{th} slot. It easy to verify that this map is well-defined. Its inverse is given by $\phi_k^{-1} : \mathbb{F}^{n-1} \rightarrow U_k$ which is given by

$$\phi_k^{-1}(y) = \phi_k^{-1}(y^1, \dots, y^{n-1}) = [y^1, \dots, 1, \dots, y^{n-1}]$$

Let $(y^1, \dots, y^{n-1}) \in \mathbb{F}^{n-1}$ and $\xi = (y^1 \cdot x, \dots, y^{k-1} \cdot x, x, y^k \cdot x, \dots, y^{n-1} \cdot x)$, so that

$$\phi_k([\xi]) = (y^1, \dots, y^{n-1})$$

which proves surjectivity. Now, take $[\xi], [\zeta] \in U_k$ with $\phi_k([\xi]) = \phi_k([\zeta])$. A simple calculation then yields

$$\zeta = \xi((\xi^k)^{-1} \zeta^k) \Rightarrow [\xi] = [\zeta]$$

which proves injectivity of ϕ_k . The continuity of ϕ_k and ϕ_k^{-1} are easy to verify by composition of continuous maps and continuity of maps out of quotient spaces. We have therefore proven that ϕ_k is an homeomorphism and since $\mathbb{F}^{n-1}$ itself is homeomorphic to $\mathbb{R}^j$ for some $j > 0$, the result is proven. $\square$

It is somewhat cumbersome to write down an explicit expression for the overlap functions, but is easy enough to generalize from a specific example:

$$\phi_2 \circ \phi_1^{-1}(y^2, \dots, y^n) = ((y^2)^{-1}, y^3(y^2)^{-1}, \dots, y^n(y^2)^{-1})$$

We now prove a result which is going to be important in our study of low-dimensional Hopf bundles, which is the relation between the 1-dimensional projective spaces and the spheres.

Theorem 7.1.1. *We have the following homeomorphisms:*

$$\mathbb{R}\mathbb{P}^1 \simeq S^1 \quad \mathbb{C}\mathbb{P}^1 \simeq S^2 \quad \mathbb{H}\mathbb{P}^1 \simeq S^4$$

Proof. We start by introducing a slight modification to the standard charts of the projective spaces. On $\mathbb{F}\mathbb{P}^1$ there are only two charts (U_1, ϕ_1) and (U_2, ϕ_2) . We define $\bar{\phi}_1 : U_1 \rightarrow \mathbb{F}$ by

$$\bar{\phi}_1([\xi]) = \overline{\phi_1([\xi])}$$

Then $(U_1, \bar{\phi}_1)$ still is a chart, but the advantage is that the overlap functions take the following form:

$$\phi_2 \circ \bar{\phi}_1^{-1}(y) = \bar{y}^{-1} = \phi_2 \circ \bar{\phi}_1^{-1}(y) \quad \forall y \in \mathbb{F} - \{0\}$$

Now let (U_S, ϕ_S) and (U_N, ϕ_N) be the stereographic projection charts on S^1, S^2 and S^4 respectively. Then their overlap functions can be written as

$$\phi_S \circ \phi_N^{-1}(y) = \bar{y}^{-1} = \phi_N \circ \phi_S^{-1}(y) \quad y \in \mathbb{F} - \{0\}$$

(if one identifies the spheres with their pendant S). Now let $(U_1, \bar{\phi}_1)$ and (U_2, ϕ_2) be the charts on $\mathbb{F}\mathbb{P}^1$ and let us consider the homeomorphisms $\phi_S \circ \phi_2 : U_2 \rightarrow U_S$ and $\phi_N^{-1} \circ \bar{\phi}_1 : U_1 \rightarrow U_N$ which satisfy

$$\phi_S \circ \phi_2|_{U_1 \cap U_2} = \phi_N^{-1} \circ \bar{\phi}_1|_{U_1 \cap U_2}$$

Indeed, if $[\xi] \in U_1 \cap U_2$, then $\phi_2([\xi]) \in \mathbb{F} - \{0\}$, on which we have

$$\begin{aligned} \bar{\phi}_1 \circ \phi_2^{-1} &= \phi_N \circ \phi_S^{-1} \\ \Rightarrow \bar{\phi}_1([\xi]) &= \phi_N \circ \phi_S^{-1} \circ \phi_2([\xi]) \\ \Rightarrow \phi_N^{-1} \circ \bar{\phi}_1([\xi]) &= \phi_S^{-1} \circ \phi_2([\xi]) \end{aligned}$$

Now, $U_1 \cup U_2 = \mathbb{F}\mathbb{P}^1$ and $U_N \cup U_S = S$ and we define:

$$\rho([\xi]) = \begin{cases} \phi_N^{-1} \circ \bar{\phi}_1([\xi]) & \text{if } [\xi] \in U_1 \\ \phi_S \circ \phi_2([\xi]) & \text{if } [\xi] \in U_2 \end{cases}$$

ρ is continuous by the Gluing lemma and trivially onto and one-to-one (since the two parts are homeomorphisms). Furthermore, the same applies to the inverse

$$\rho^{-1}(y) = \begin{cases} \bar{\phi}_1^{-1} \circ \phi_N(y) & \text{if } y \in U_N \\ \phi_2^{-1} \circ \phi_S(y) & \text{if } y \in U_S \end{cases}$$

So that ρ is the sought homeomorphism. □

Corollary 7.1.1. *The above homeomorphisms are diffeomorphisms, which can be rewritten as*

$$\rho([\xi^1, \xi^2]) = (2\xi^1\xi^2, |\xi^1|^2 - |\xi^2|^2)$$

Proof. Follows from a simple computation. □

7.2 Projective spaces as locally trivial bundles: Hopf bundles

7.2.1 The real Hopf bundle

Let $\mathcal{P} : S^{n-1} \rightarrow \mathbb{R}\mathbb{P}^{n-1}$ be the quotient map $\mathcal{P}(x) = [x]$ for every $x \in S^{n-1}$. We consider the open hemispheres,

$$U_k^+ = \{x \in S^{n-1} | x^k > 0\} \quad U_k^- = \{x \in S^{n-1} | x^k < 0\} \quad \forall k = 1, \dots, n$$

They are open, satisfy $U_k^+ \cap U_k^- = \emptyset$ and cover S^{n-1} . The restriction of $\mathcal{P}$ is clearly continuous and also one-to-one since $\mathcal{P}$ identifies antipodal points.

We claim that each $\mathcal{P}|_{U_k^\pm}$ is an open map. Indeed, let U be an open set in $U_k^\pm$, then

$$\mathcal{P}^{-1}((\mathcal{P}|_{U_k^\pm})(U)) = \mathcal{P}^{-1}(\mathcal{P}(U)) = U \cup (-U)$$

where $-U = \{-x : x \in U\}$. By the union of open set, this is open, so that $\mathcal{P}|_{U_k^\pm}$ is open. Thus, every $\mathcal{P}|_{U_k^\pm}$ is a homeomorphism of $U_k^\pm$ onto $V_k = \mathcal{P}(U_k^+) = \mathcal{P}(U_k^-)$. Moreover, we have

$$\mathcal{P}^{-1}(V_k) = U_k^+ \cup U_k^-$$

which are disjoint and each mapped homeomorphically onto V_k by $\mathcal{P}$. We can therefore define the maps

$$\Phi_k : V_k \times \mathbb{Z}_2 \rightarrow \mathcal{P}^{-1}(V_k) = U_k^+ \cup U_k^-$$

by

$$\Phi_k([x], 1) = \left(\mathcal{P}|_{U_k^+}\right)^{-1}([x]) \quad \Phi_k([x], -1) = \left(\mathcal{P}|_{U_k^-}\right)^{-1}([x])$$

for all $[x] \in V_k$. We now have

$$V_k \times \mathbb{Z}_2 = V_k \times \{1\} \cup V_k \times \{-1\} \quad \text{where} \quad (V_k \times \{1\}) \cap (V_k \times \{-1\}) = \emptyset$$

The maps $\Phi_k|_{V_k \times \{1\}}$ and $\Phi_k|_{V_k \times \{-1\}}$ are homeomorphisms simply by composition of homeomorphisms.

We have that $(V_k \times \{1\}) \cup (V_k \times \{-1\}) = V_k \times \mathbb{Z}_2$ and the intersection is empty. Furthermore, the two Φ_k are continuous, so that by the gluing lemma, the entire Φ_k is continuous. Exactly the same argument applies to the inverse. Finally, we also have

$$\mathcal{P} \circ \Phi_k([x], \pm 1) = \mathcal{P}(\mathcal{P}^{-1}([x]) \cap U_k^\pm) = [x]$$

So we have proven that $(S^{n-1}, \mathbb{RP}^{n-1}, \mathcal{P}, \mathbb{Z}_2)$ is a locally trivial bundle.

7.2.2 The complex and quaternionic Hopf bundle

In this section, we prove that the complex and quaternionic projective spaces (so that in this section $\mathbb{F} = \mathbb{C}, \mathbb{H}$ and their projection maps are locally trivial bundles. We prove both cases together. To do so, we will denote the bundle spaces S , i.e. $S = S^{2n-1}$ in the complex case and $S = S^{4n-1}$ in the quaternionic case, and S_F the fibers, i.e. $S_F = S^1$ in the complex case and $S_F = S^3$ in the quaternionic case. Furthermore, the fibers S_F are equipped with their natural product structure, inherited from complex and quaternionic multiplication respectively.

For every $k = 1, \dots, n$, we define

$$V_k = \{[\xi] \in \mathbb{FP}^{n-1} | z^k \neq 0\}$$

as in the real case. Since the V_k cover $\mathbb{FP}^{n-1}$, it suffices to define the homeomorphisms

$$\Phi_k : V_k \times S_F \rightarrow \mathcal{P}^{-1}(V_k)$$

which satisfy

$$\mathcal{P} \circ \Phi_k([\xi], y) = [\xi]$$

for all $([\xi], y) \in V_k \times S_F$. We now fix $[\xi] \in V_k$ and $y \in S_F$. We then define

$$\Phi_k([\xi], y) = \left(z^1 (z^k)^{-1} |z^k| y, \dots, z^n (z^k)^{-1} |z^k| y\right)$$

where $(z^1, \dots, z^n)$ is any point in $\mathcal{P}^{-1}([\xi])$. Indeed, let $[\xi] = [z^1, \dots, z^n] = [w^1, \dots, w^n]$ and $y \in S_F$, then we have $z = w\alpha$ where $\alpha \in \mathbb{F} - \{0\}$ and $|\alpha| = 1$, so that

$$\begin{aligned} \left(z^1 (z^k)^{-1} |z^k| y, \dots, z^n (z^k)^{-1} |z^k| y\right) &= \left(w^1 \alpha (w^k \alpha)^{-1} |\alpha w^k| y, \dots, w^n \alpha (w^k \alpha)^{-1} |\alpha w^k| y\right) \\ &= \left(w^1 (w^k)^{-1} |w^k| y, \dots, w^n (w^k)^{-1} |w^k| y\right) \end{aligned}$$

Furthermore, since $(z^k)^{-1} |z^k| y \in \mathbb{F} - \{0\}$ and $|(z^k)^{-1} |z^k| y| = 1$, we have

$$\mathcal{P} \circ \Phi_k([\xi], y) = [z^1 (z^k)^{-1} |z^k| y, \dots, z^n (z^k)^{-1} |z^k| y] = [\xi]$$

We now define the map $\Psi_k : \mathcal{P}^{-1}(V_k) \rightarrow V_k \times S_F$ by

$$\Psi_k(\xi) = \Psi_k(z^1, \dots, z^n) = ([\xi], |z_k|^{-1} z^k) = (\mathcal{P}(\xi), |z_k|^{-1} z^k)$$

which is the inverse of Φ_k since

$$\Phi_k \circ \Psi_k(\xi) = \left(z^1 (z^k)^{-1} |z^k| |z^k|^{-1} z^k, \dots, z^n (z^k)^{-1} |z^k| |z^k|^{-1} z^k\right) = \xi$$

and

$$\Psi_k \circ \Phi_k([\xi], y) = ([\xi], y)$$

by simply unfolding the definitions.

Ψ_k is clearly continuous since both component functions are continuous.

One also easily concludes that Φ_k is continuous.

Thus the (V_k, Φ_k) are the required trivializations.

We have therefore two new families of locally trivial bundles called the Hopf bundles:

1. $(S^{2n-1}, \mathbb{CP}^{n-1}, \mathcal{P}, S^1)$ or $S^1 \rightarrow S^{2n-1} \rightarrow \mathbb{CP}^{n-1}$
2. $(S^{4n-1}, \mathbb{HP}^{n-1}, \mathcal{P}, S^3)$ or $S^3 \rightarrow S^{4n-1} \rightarrow \mathbb{HP}^{n-1}$

We are going to be particularly interested in the Hopf bundles for $n = 2$ because of the relation between the one dimensional projective spaces and spheres, which yield the locally trivial bundles

$$S^1 \rightarrow S^3 \rightarrow S^2 \quad \text{and} \quad S^3 \rightarrow S^7 \rightarrow S^4$$

7.3 Hopf bundles as principal bundles

7.3.1 The real case

Consider the locally trivial bundle $(S^{n-1}, \mathbb{RP}^{n-1}, \mathcal{P}, \mathbb{Z}_2)$. We then define the natural action of $\mathbb{Z}_2$ on S^{n-1} by

$$p \cdot g = (x^1 g, \dots, x^n g)$$

for all $p = (x^1, \dots, x^n) \in S^{n-1}$ and all $g \in \mathbb{Z}_2$. Since $\mathcal{P}$ identifies antipodal points, we trivially have

$$\mathcal{P}(p \cdot g) = \mathcal{P}(p)$$

Furthermore, if we define $\Psi_k = \Phi_k^{-1}$ where Φ_k are the trivializations defined in section 7.2.1, we then get

$$\Psi_k(p) = (\mathcal{P}(p), \psi_k(p))$$

where $\psi_k : \mathcal{P}^{-1}(V_k) \rightarrow \mathbb{Z}_2$ takes value 1 on U_k^+ and -1 on U_k^- . Indeed,

$$\begin{aligned} \Psi_k \circ \Phi_k([x], g) &= \begin{cases} \Psi_k \left(\left(\mathcal{P}|_{U_k^+} \right)^{-1}([x]) \right) & \text{if } g = 1 \\ \Psi_k \left(\left(\mathcal{P}|_{U_k^-} \right)^{-1}([x]) \right) & \text{if } g = -1 \end{cases} \\ &= ([x], g) \end{aligned}$$

since on $U_k^\pm$ the projection is a homeomorphism and in the same way, one proves that $\Phi_k \circ \Psi_k = \text{id}$. In particular, we also have

$$\psi_k(p \cdot 1) = \psi_k(p)1 \quad \psi_k(p \cdot (-1)) = -\psi_k(p) = \psi_k(p)(-1)$$

We have therefore proven that S^{n-1} is a principal $\mathbb{Z}_2$ -bundle over $\mathbb{RP}^{n-1}$.

7.3.2 The complex and quaternionic cases

As in section 7.2.2, we treat the complex and quaternionic cases together and we keep the notations we established in that section. To comply with the definition, we identify $S^1 \simeq U(1)$ and $S^3 \simeq Sp(1)$ which are both Lie groups. We define the corresponding natural action on S by

$$p \cdot g = (\xi^1 g, \dots, \xi^n g)$$

where $\xi = (\xi^1, \dots, \xi^n) \in S$ and $g \in S_F$. This action trivially satisfies the fiber invariance condition

$$\mathcal{P}(p \cdot g) = \mathcal{P}(p)$$

In section 7.2.2, we also defined the homeomorphisms (diffeomorphisms) Ψ_k which were given by

$$\Psi_k(p) = (\mathcal{P}(p), \psi_k(p)) \quad \text{where} \quad \psi_k(p) = |z^k|^{-1} z^k$$

For any $g \in S_F$, we have

$$\psi_k(p \cdot g) = |z^k g|^{-1} (z^k g) = (|z^k|^{-1} z^k) g = \psi_k(p) g$$

since $|g| = 1$.

So we have proven that S^{2n-1} is a principal $U(1)$ -bundle over $\mathbb{CP}^{n-1}$ and that S^{4n-1} is a principal $Sp(1)$ -bundle over $\mathbb{HP}^{n-1}$.

Let us now write down the transition functions and the canonical cross-sections for these principal bundles. If $x \in V_i \cap V_j$ and $p = (z^1, \dots, z^n)$ any point in $\mathcal{P}^{-1}(x)$ in which case we obtain

$$g_{ij}(x) = |z^j|^{-1} z^j (z^i)^{-1} |z^i|$$

Let (V_k, Ψ_k) be the standard trivializations of these principal bundles. For every $k = 1, \dots, n$, the corresponding canonical cross-section $s_k \equiv s_{V_k} : V_k \rightarrow \mathcal{P}^{-1}(V_k)$ is given by

$$s_k(x) = \Psi_k^{-1}([\xi^1, \dots, \xi^n], e) = \left(\xi^1 (\xi^k)^{-1} |\xi^k|, \dots, \xi^n (\xi^k)^{-1} |\xi^k| \right)$$

for any $(\xi^1, \dots, \xi^n) \in \mathcal{P}^{-1}(x)$. Thus,

$$s_k(x) = \left(\xi^1 (\xi^k)^{-1} |\xi^k|, \dots, |\xi^k|, \dots, \xi^n (\xi^k)^{-1} |\xi^k| \right)$$

where $|\xi^k|$ is in the k^{th} slot

In particular, for $n = 2$, we obtain S^3 as a principal $U(1)$ -bundle over S^2 and S^7 as a principal $Sp(1)$ -bundle over S^4 . We point out that by the classification theorem for principal bundles over spheres, we have that the equivalence classes of principal $U(1)$ -bundles over S^2 are in one-to-one correspondence with the elements of $\pi_1(U(1)) \simeq \pi_1(S^1) \simeq \mathbb{Z}$ and the equivalence classes of principal $Sp(1)$ -bundles over S^4 are in one-to-one correspondence with the elements of $\pi_3(Sp(1)) \simeq \pi_3(S^3) \simeq \mathbb{Z}$.

8 Connections on Hopf bundles

8.1 Magnetic monopole connection

In this section, we exhibit a connection on S^3 viewed as a principal $U(1)$ -bundle over S^2 described in section 7.3.2. For convenience, we will identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and therefore

$$S^3 = \{(z^1, z^2) \in \mathbb{C}^2 \mid |z^1|^2 + |z^2|^2 = 1\}$$

We then parametrize the z_i in the standard polar way by $z_i = r_i e^{i\xi_i}$. Furthermore, since $r_1^2 + r_2^2 = 1$ and $r_i > 0$, there exists a $\theta \in [0, \pi]$ such that

$$r_1 = \cos\left(\frac{\phi}{2}\right) \quad r_2 = \sin\left(\frac{\phi}{2}\right)$$

We have thus parametrized the 3-sphere as

$$S^3 = \left\{ \left(\cos\left(\frac{\phi}{2}\right) e^{i\xi_1}, \sin\left(\frac{\phi}{2}\right) e^{i\xi_2} \right) \mid \phi \in [0, \pi], \xi_1, \xi_2 \in \mathbb{R} \right\}$$

The diffeomorphism from Corollary 7.1.1 of $\mathbb{CP}^1$ onto $S^2 \subseteq \mathbb{C} \times \mathbb{R}$ can be written as

$$\rho([z^1, z^2]) = (z^1 \bar{z}^2 + \bar{z}^1 z^2, -iz^1 \bar{z}^2 + i\bar{z}^1 z^2, |z^1|^2 - |z^2|^2)$$

Furthermore, it is easy to see that this diffeomorphism carries U_1 onto U_N and U_2 onto U_S . To avoid complicating the notation unnecessarily, we will suppress the diffeomorphism ρ and write

$\rho \circ \mathcal{P} : S^3 \rightarrow S^2$ as $\mathcal{P}$ and thereby regard the Hopf projection map $\mathcal{P}$ as a map of S^3 onto S^2 given by

$$\mathcal{P}(z^1, z^2) = (z^1 \bar{z}^2 + \bar{z}^1 z^2, -iz^1 \bar{z}^2 + i\bar{z}^1 z^2, |z^1|^2 - |z^2|^2)$$

In terms of real coordinates given by $(z^1, z^2) = (x^1 + y^1 i, x^2 + y^2 i)$, the projection maps becomes

$$\mathcal{P}(x^1, y^1, x^2, y^2) = (2x^1 x^2 + 2y^1 y^2, 2x^2 y^1 - 2x^1 y^2, (x^1)^2 + (y^1)^2 - (x^2)^2 - (y^2)^2)$$

which in turn implies that the projection map in terms of the parameters ϕ, ξ_1, ξ_2 which describe S^3 is given by

$$\mathcal{P}(\phi, \xi_1, \xi_2) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

where $\theta = \xi_1 - \xi_2$. Notice that this is exactly the point of S^2 with standard spherical coordinates $(\phi, \theta) = (\phi, \xi_1 - \xi_2)$, which in particular motivates our subsequent use of spherical coordinates on S^2 .

Before proceeding, we make one remark on spherical coordinates. Standard spherical coordinates on $\mathbb{R}^3$ are given by the homeomorphism

$$(\rho, \phi, \theta) \rightarrow (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

of $(0, \infty) \times (0, \pi) \times (0, 2\pi)$ onto the open set $U = \mathbb{R}^3 - \{x = 0, y \leq 0\}$. The point of this is that if you want to cover all of $\mathbb{R}^3 - \{0\}$ you need at least four charts. These charts differ only by the domain on which they are defined. Putting $\rho = 1$, one obtains a coordinate slices, which are the standard spherical coordinate submanifold charts on S^2 . Because these charts only differ in their ranges, the sensible and usual thing to do is to simply refer to (ϕ, θ) as the standard spherical coordinates on S^2 .

Let us also exhibit the transition functions of this bundle in terms of the above parameters. We have

$$\begin{aligned} g_{SN}(x) &= \frac{z^2/|z^2|}{z^1/|z^1|} & \text{and} & & g_{NS}(x) &= \frac{z^1/|z^1|}{z^2/|z^2|} \\ \Rightarrow g_{SN}(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) &= e^{-i\theta} \\ \Rightarrow g_{NS}(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) &= e^{i\theta} \end{aligned}$$

where again $\theta = \xi_1 - \xi_2$.

Since the standard trivializing neighborhoods are carried onto $U_{N/S}$ by Corollary 7.1.1, we have now that the trivializing neighborhoods are U_S and U_N so that the associated cross-sections are $s_N : U_N \rightarrow \mathcal{P}^{-1}(U_N)$ and $s_S : U_S \rightarrow \mathcal{P}^{-1}(U_S)$. We have shown in section 7.3.2 that the canonical cross-sections have the form:

$$\begin{aligned} s_N(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) &= \left(|z^1|, \frac{z^2 |z^1|}{z^1} \right) = \left(\cos \frac{\phi}{2}, \sin \frac{\phi}{2} e^{-i\theta} \right) \\ s_S(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) &= \left(\frac{z^1 |z^2|}{z^2}, |z^2| \right) = \left(\cos \frac{\phi}{2} e^{i\theta}, \sin \frac{\phi}{2} \right) \end{aligned}$$

identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ as usual, i.e. $(x^1 + iy^1, x^2 + iy^2) \rightarrow (x^1, y^1, x^2, y^2)$ and letting $i : S^3 \hookrightarrow \mathbb{R}^4$ be the inclusion map, we obtain:

$$\begin{aligned} i \circ s_N(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) &= \left(\cos \frac{\phi}{2}, 0, \sin \frac{\phi}{2} \cos \theta, -\sin \frac{\phi}{2} \sin \theta \right) \\ i \circ s_S(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) &= \left(\cos \frac{\phi}{2} \cos \theta, \cos \frac{\phi}{2} \sin \theta, \sin \frac{\phi}{2}, 0 \right) \end{aligned}$$

Now we consider the 1-form $\tilde{\Theta}$ on $\mathbb{R}^4$ given in the standard global chart by

$$\tilde{\Theta} = -x^2 dx^1 + x^1 dx^2 - x^4 dx^3 + x^3 dx^4$$

We then define

$$\omega = i(i^*\tilde{\Theta})$$

We are going to prove that this is the connection form on this bundle. To do, we start by calculating the local gauge fields by pulling ω back by the canonical cross-sections. Two trivial calculations yield:

$$\begin{aligned} s_N^*\Theta &= (i \circ s_N)^*\tilde{\Theta} \\ \Rightarrow \mathcal{A}_N \equiv (s_N\Theta)^*(\phi, \theta) &= -\frac{1}{2}i(1 - \cos \phi)d\theta \\ \Rightarrow \mathcal{A}_S \equiv (s_S\Theta)^*(\phi, \theta) &= \frac{1}{2}i(1 + \cos \phi)d\theta \end{aligned}$$

In particular, this implies that on $U_S \cap U_N$, we have

$$\mathcal{A}_N - \mathcal{A}_S = -id\theta$$

For this condition to resemble the local gauge transformations, we rewrite it as:

$$\begin{aligned} \mathcal{A}_N &= e^{i\theta}\mathcal{A}_S e^{-i\theta} + e^{i\theta}de^{-i\theta} \\ \Rightarrow \mathcal{A}_N &= g_{SN}^{-1}\mathcal{A}_S g_{SN} + g_{SN}d(g_{SN}^{-1}) \end{aligned}$$

Now, Lemma 4.1.2 and Theorem 4.1.1 then imply that ω is a connection 1-form on S^3 with local gauge field $\mathcal{A}_N$ and $\mathcal{A}_S$.

8.2 BPST instanton connection

In this section, we compute the natural connection on S^7 viewed as a principal $Sp(1)$ -bundle over S^4 .

We start by establishing the notation. For this calculation, it is very convenient to work with quaternions, so that again we identify

$$S^7 = \{(q^1, q^2) \in \mathbb{H}^2 \mid |q^1|^2 + |q^2|^2 = 1\}$$

We also identify the Lie algebra of $Sp(1)$ with $\text{Im}\mathbb{H}$. The trivializations are given by

$$\begin{aligned} \Psi_k(p) &= (\mathcal{P}(p), \psi_k(p)) \quad \text{where} \quad \psi_k(p) = |q^k|^{-1}q^k \\ \Phi_1([q^1, q^2], y) &= (|q^1|y, q^2(q^1)^{-1}|q^1|y) \\ \Phi_2([q^1, q^2], y) &= (q^1(q^2)^{-1}|q^2|y, |q^2|y) \end{aligned}$$

The canonical cross-sections have been given before. Furthermore, V_1 and V_2 are also the standard coordinate neighborhoods for $\mathbb{H}\mathbb{P}^1$. The corresponding diffeomorphisms $\phi_k : V_k \rightarrow \mathbb{R}^4$ are given by

$$\begin{aligned} \phi_1([q^1, q^2]) &= q^2(q^1)^{-1} & \& \quad \phi_2([q^1, q^2]) = q^1(q^2)^{-1} \\ \phi_1^{-1}(q) = [1, q] &= \left[(1 + |q|^2)^{-\frac{1}{2}}, q(1 + |q|^2)^{-\frac{1}{2}}\right] & \& \quad \phi_2^{-1}(q) = [q, 1] = \left[q(1 + |q|^2)^{-\frac{1}{2}}, (1 + |q|^2)^{-\frac{1}{2}}\right] \end{aligned}$$

where the second form is more convenient because it actually belongs to S^7 . The local coordinate expressions of the canonical cross-sections are then given by

$$(s_1 \circ \phi_1^{-1})(q) = (1 + |q|^2)^{-\frac{1}{2}}(1, q) \quad \& \quad (s_2 \circ \phi_2^{-1})(q) = (1 + |q|^2)^{-\frac{1}{2}}(q, 1)$$

We now define the $\text{Im}\mathbb{H}$ valued 1-form on S^7 by first defining

$$\tilde{\omega} = \text{Im}(\bar{q}^1 dq^1 + \bar{q}^2 dq^2)$$

on $\mathbb{H}^2$ and then ω by restricting to S^7 , i.e.

$$\omega = i^*\tilde{\omega}$$

We then obtain

$$\omega_p(\mathbf{v}) = \omega_{(p^1, p^2)}(v^1, v^2) = \text{Im}(\bar{p}^1 v^1 + \bar{p}^2 v^2)$$

where $p \in S^7 \subseteq \mathbb{H}^2$ and $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ which we identified with $(v^1, v^2) \in \mathbb{H}$ using the fact that $dq^i(\mathbf{v}^1, \mathbf{v}^2) = v^i$ where v^i is the element of $\mathbb{H}$ corresponding to $\mathbf{v}^i$ under the canonical isomorphism of tangent spaces of vector spaces.

We now show that ω satisfies the properties of connections. We start by right equivariance under the standard action σ of $Sp(1)$ on S^3 . We have that if $\mathbf{v} = (v^1, v^2) \in T_{p \cdot g^{-1}}(S^7)$, then

$$(\sigma_g)_{*p \cdot g^{-1}}(\mathbf{v}) = (v^1 g, v^2 g)$$

We therefore get

$$\omega_p((\sigma_g)_{*p \cdot g^{-1}}(\mathbf{v})) = \text{Im}(\bar{p}^1 v^1 g + \bar{p}^2 v^2 g)$$

On the other hand, we have

$$\begin{aligned} \omega_{p \cdot g^{-1}}(\mathbf{v}) &= \text{Im}(\overline{p^1 g^{-1}} v^1 + \overline{p^2 g^{-1}} v^2) \\ &= \text{Im}(g \bar{p}^1 v^1 + g \bar{p}^2 v^2) \quad \text{since } \overline{g^{-1}} = g \\ \Rightarrow ad_{g^{-1}}(\omega_{p \cdot g^{-1}}(\mathbf{v})) &= g^{-1} \text{Im}(g \bar{p}^1 v^1 + g \bar{p}^2 v^2) g \\ &= \text{Im}(\bar{p}^1 v^1 g + \bar{p}^2 v^2 g) \end{aligned}$$

where in the last line, we used that if $g \in Sp(1)$ and $h \in \mathbb{H}$, we have

$$g^{-1} \text{Im}(h) g = \text{Im}(g^{-1} h g)$$

We have thus proven

$$ad_{g^{-1}}(\omega_{p \cdot g^{-1}}(\mathbf{v})) = \omega_p((\sigma_g)_{*p \cdot g^{-1}}(\mathbf{v}))$$

which proves right equivariance under the standard action.

We now show that ω acts trivially on fundamental vector fields. Let $A \in \text{Im}\mathbb{H}$. We have

$$A^\#(p) = (p \cdot \exp(tA))'(0) = (p^1 A, p^2 A)$$

so that

$$\omega_p(A^\#(p)) = \text{Im}(\bar{p}^1 p^1 A + \bar{p}^2 p^2 A) = \text{Im}A = A$$

since $p \in S^7$ and $A \in \text{Im}\mathbb{H}$.

We have therefore proven that ω is a connection on S^7 viewed as a principal $Sp(1)$ -bundle over S^4 .

For completeness, we now exhibit the local gauge fields. For this last part, we do not write down the computations explicitly since they are quite lengthy, but rather trivial even if one needs to be a bit careful. In the end, it is a simple unfolding of definitions. In order to have some coordinates in which to write things in, we furthermore pull the local gauge fields back to $\mathbb{H}$ via the standard charts on $\mathbb{H}\mathbb{P}^1$. More explicitly, for each $q \in \mathbb{H}$, each $v \in T_q(\mathbb{H})$ and each $k = 1, 2$, we have

$$\left((s_k \circ \phi_k^{-1})^* \omega \right)_q(v) = (s_k^* \omega)_{\phi_k^{-1}(q)} \left((\phi_k^{-1})_{*q}(v) \right)$$

The computation then yields

$$\begin{aligned} ((s_k \circ \phi_k)^* \omega)_q(v_q) &= \text{Im} \left(\frac{\bar{q}}{1 + |q|^2} v_q \right) \\ \Rightarrow (s_k \circ \phi_k^{-1})^* \omega &= \text{Im} \left(\frac{\bar{q}}{1 + |q|^2} dq \right) \end{aligned}$$

where $q \in \mathbb{H}$ fixed and v the quaternion corresponding to $v_q \in T_q(\mathbb{H})$ fixed. In last line, we regard $\bar{q}$ and $|q|^2$ as functions on $\mathbb{H}$ and with the standard coordinates, one then has $dq(v_q) = v$. This

notation can be a bit confusing since for $k = 1$ or $k = 2$, the q refers to different coordinate patches. A bit more precisely, we have

$$\begin{aligned}(s_1^*\omega)_p \mathbf{X} &= \left((s_1 \circ \phi_1^{-1})^* \omega \right)_{\phi_1(p)} ((\phi_1)_* \mathbf{X}) = \operatorname{Im} \left(\frac{\overline{\phi_1(p)} v}{1 + |\phi_1(p)|^2} \right) \quad \text{for } p \in V_1, v = dq((\phi_1)_* \mathbf{X}) \\ (s_2^*\omega)_p \mathbf{X} &= \left((s_2 \circ \phi_2^{-1})^* \omega \right)_{\phi_2(p)} ((\phi_2)_* \mathbf{X}) = \operatorname{Im} \left(\frac{\overline{\phi_2(p)} w}{1 + |\phi_2(p)|^2} \right) \quad \text{for } p \in V_2, w = dq((\phi_2)_* \mathbf{X})\end{aligned}$$

where $\mathbf{X} \in T_p(S^4)$.

Finally, using the structure equation and again a lengthy but trivial, calculation one finds

$$\Omega = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = \frac{1}{1 + |q|^2} d\bar{q} \wedge dq$$

where $\mathcal{A} = \operatorname{Im} \left(\frac{\bar{q}}{1 + |q|^2} dq \right)$.

9 Magnetic monopoles

9.1 à la physicist

During the elaboration of the laws of electric and magnetic phenomena, which culminated in Maxwell's laws, it was well established that there exists electric charges (or monopoles), whereas magnetic monopoles do not exist. Experimentally, this is still the case today. But from a theoretical viewpoint, it is interesting to speculate about the possible existence of magnetic monopoles, which was first attempted by Dirac. This attempt led him to a rather remarkable relation, called Dirac's charge quantization condition.

In this section, we assume a minimal amount of knowledge in electromagnetism and quantum mechanics. Let us suppose that a magnetic monopole exists. Coulomb's law would then tell us that this particle determines the following fields

$$\mathbf{E} = 0 \quad \mathbf{B} = \frac{g}{\rho^2} \hat{e}_\rho$$

where $\hat{e}_\rho$ is the unit radial vector. Indeed, on $\mathbb{R}^3 - \{0\}$, these fields satisfy

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \wedge \mathbf{B} = 0$$

It also well known that how one generally proceeds is by defining a scalar and a vector potential which satisfy

$$\mathbf{B} = \nabla \wedge \mathbf{A} \quad \mathbf{E} = -\nabla V$$

(We do not consider any time dependence). The introduction of the vector potential was originally a purely mathematical trick, because one can show that the equations which the potentials satisfy (in a certain gauge, generally the Coulomb gauge) are simple wave equations with sources. It was not until the advent of quantum mechanics that it was realized that the vector potential actually has a physical significance.

For now, let us continue our computation. It is easy to see that there cannot exist a globally defined vector potential. Indeed,, if one calculates the flux of the magnetic field through some sphere S one obtains:

$$\Phi = \iint_S \mathbf{B} \cdot d\mathbf{S} = \iint_S \nabla \wedge \mathbf{A} \cdot d\mathbf{S} = 4\pi g$$

but using Stokes theorem, we obtain:

$$\iint_S \nabla \wedge \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{r} - \oint_C \mathbf{A} \cdot d\mathbf{r} = 0$$

which is a contradiction. One can state this more generally. Indeed, while being simply connected is a sufficient condition for the existence of a scalar potential, it is not sufficient for the existence of a vector potential. We have the following theorem.

Theorem 9.1.1. *Let U be an open, simply connected subset of $\mathbb{R}^3$ and $\mathbf{F}$ a smooth vector field on U . If $\nabla \cdot \mathbf{F} = 0$ on U and if the second homotopy group $\pi_2(U)$ is trivial, then there exists a smooth vector field $\mathbf{A}$ on U such that*

$$\mathbf{F} = \nabla \wedge \mathbf{A}$$

The way to solve this problem is abandon the notion of a globally defined vector potential. Indeed what one can do is simply define two separate regions of space on which there exists a globally defined vector potential and which cover the entire space. For example, in our case, we define

$$\begin{aligned} U_+ &= \mathbb{R}^3 - Z_+ & \text{where } Z_+ &= \{(0, 0, z) \in \mathbb{R}^3 | z \geq 0\} \\ U_- &= \mathbb{R}^3 - Z_- & \text{where } Z_- &= \{(0, 0, z) \in \mathbb{R}^3 | z \leq 0\} \end{aligned}$$

We then have $U_+ \cup U_- = \mathbb{R}^3 - \{0\}$ and $U_+ \cap U_- = \mathbb{R}^3 - \{(0, 0, z) \in \mathbb{R}^3 | z \in \mathbb{R}\}$. We also define

$$\begin{aligned} \mathbf{A}_+(\rho, \phi, \theta) &= \frac{g}{\rho \sin \phi} (1 - \cos \phi) \hat{e}_\theta & \text{on } U_+ \\ \mathbf{A}_-(\rho, \phi, \theta) &= \frac{g}{\rho \sin \phi} (1 + \cos \phi) \hat{e}_\theta & \text{on } U_- \end{aligned}$$

Here comes the crucial part: On $U_+ \cap U_-$, we have

$$\mathbf{A}_+ - \mathbf{A}_- = \left(\frac{2g}{\rho \sin \phi} \right) \hat{e}_\theta = \nabla(2g\theta)$$

where in the last equality we used the expression of the gradient in spherical coordinates. The reason why this is so important is because the Maxwell equations have a well-known symmetry, gauge symmetry, which basically states that two vector potentials which differ by a gradient are physically equivalent.

Up until now, this could be considered nothing more than a mathematical curiosity, but things get interesting when you put quantum mechanics into the mix. Indeed, the Schrödinger equation for the wave function of this system is given by

$$\frac{1}{2m} \left(-i\hbar \nabla - \frac{e}{c} \mathbf{A} \right)^2 \psi(\mathbf{r}) = E\psi(\mathbf{r})$$

It takes a few lines to prove how the wavefunction transforms under gauge transformations:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \Omega \Rightarrow \psi \rightarrow e^{iq\Omega} \psi$$

If we denote by ψ_+ and ψ_- the wavefunctions on the two domains, the previous transformation law tells us that

$$\psi_+ = e^{i2qg\theta} \psi_-$$

But on $U_+ \cap U_-$ both ψ_+ and ψ_- are single-valued, so that in particular the transformation $\theta \rightarrow \theta + 2\pi$ must leave the wavefunctions invariant. This in turn then implies that

$$e^{iqg4\pi} = 1 \Rightarrow qg = \frac{1}{2}n \quad \text{where } n \in \mathbb{Z}$$

At this point, one can intuitively sense the relation between this quantization condition and the topology of the underlying $U(1)$ -bundle over S^2 , but to make this concrete, one needs a lot more tools.

9.2 à la mathematician

Create the other bundles by since we know the classification by $\mathbb{Z} g_{NS}^n$ where $n \in \mathbb{Z}$ then what are the $\mathcal{A}$

10 Where to go from here?

To be able to use all of what has been done up until in the context of “real” world physics, more precisely gauge field theories, one needs to introduce several other concepts. First of all, since we are doing physics, we need to know the set of equations describing the system. In field theories, these equations arise from a minimization problem called the least action principle. Its most simple form for a free bosonic theory is to define the Yang-Mills action of $\mathcal{A}$ denoted $\mathcal{YM}(\mathcal{A})$ by

$$\mathcal{YM}(\mathcal{A}) = \int_{\mathbb{R}^4} \|\mathcal{F}(q)\|$$

In particular, the absolute minima (which are called instantons) are of big interest because at the end of long road, the study of these objects leads to the “Donaldson-inspired revolution in low dimensional topology. On this road, there also is the study of moduli spaces which we defined in the bulk of the text, in particular of their topological properties.

The Yang-Mills action presented above is of great interest in mathematics, but from a point view of physics it is rather dull, because only describes bosons (force carriers) and their self-interaction without any interaction with matter. To incorporate matter into the mix, one needs to introduce the concept of associated fiber bundles. The idea is simple enough. A quantum mechanical particle has some internal structure which is described by a Lie group G and its wavefunction takes values in some vector space $\mathcal{V}$. The gauge field, which represents the interactions in the broad sense, is represented by a connection on a principal G -bundle. The response of the wavefunction at each point to a gauge transformation will then be modeled by a left action of G on $\mathcal{V}$: enters representation theory. Very roughly, to construct these associated fiber bundles, one replaces in a meaningful way the fibers (diffeomorphic to G) of the principal G -bundle by copies of $\mathcal{V}$. Finally, one can then introduce a gauge invariant differentiation process for such wave functions called the covariant derivative.

The reader may also have noticed that in this report, we hinted at a relation between the $U(1)$ -bundles over S^2 and the monopoles, but this relation came more from the fact that it hit us right in face with the Dirac quantization condition and the one-to-one correspondence of the $U(1)$ bundles with $\mathbb{Z}$. To actually make this relation concrete, one needs to introduce a another concept which are Chern classes.

References

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